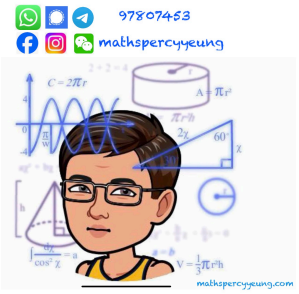
S.4 Mathematics
Ch 8 Logarithmic Functions



Name: _____ Class: _____

1. Section 8.1 Common Logarithms: Definition of Common Logarithms

1. Please refer to the textbook P.2 – 4.
2. Video from Identity: https://www.youtube.com/watch?v=cEprpTdnmfs&feature=emb_logo

I. Pre-lesson

Key Points

2. Definition of Common Logarithms

Let x be a positive number.

(a) If $x = 10^y$, then $y = \log x$.
y is called the **logarithm** of x with base 10 (or the common logarithm of x).

(b) If $y = \log x$, then $x = 10^y$.

(c) $\log 10^y = y$

Note: For $x \leq 0$, $\log x$ is undefined.
e.g. $\log 0$ and $\log(-3)$ are undefined.

1. Complete the following table.

	Exponential form ($x = 10^y$)	Logarithmic form ($\log x = y$)
(a)	$1000 = 10^3$	
(b)		$\log 100\,000 = 5$
(c)	$0.001 = 10^{-3}$	
(d)	$8 = 10^a$	
(e)		$\log b = -7$

2. Find the values of the following common logarithms without using a calculator.

(a) $\log 10\,000\,000$ **(b)** $\log 0.000\,001$ **(c)** $\log\left(\frac{1}{10}\right)$

II. Teaching example and classwork

3. Find the values of the following common logarithms without using a calculator.

(a) $\log 10^{7}$ **(b)** $\log 10^{-9}$ **(c)** $\log 10^{3.4}$

4. **(a)** $\log 100\,000\,000$ **(b)** $\log 0.000\,000\,1$ **(c)** $\log\left(\frac{1}{10\,000}\right)$

5. **(a)** $\log\sqrt{100\,000}$ **(b)** $\log\sqrt{0.001}$ **(c)** $\log\sqrt[4]{100}$

6. Find the values of the following common logarithms by using a calculator. (Give your answers correct to 3 significant figures.)

(a) $\log 90 =$ _____ (cor. to 3 sig. fig.) **(b)** $\log 0.7 =$ _____ (cor. to 3 sig. fig.)

(c) $\log\left(\frac{1}{90}\right) =$ _____ (cor. to 3 sig. fig.) **(d)** $\log 0.7^{10} =$ _____ (cor. to 3 sig. fig.)

7. Find the values of the following unknowns correct to 3 significant figures.

(a) $10^{y} = 6$ **(b)** $10^{y} = \frac{2}{3}$

(c) $\log x = 0.1$ **(d)** $\log x = -1.8$

B. Section 8.1 Common Logarithms: Properties of Common Logarithms

1. Please refer to the textbook P.4 – 9.
2. Video from Identity: https://www.youtube.com/watch?v=KgekT4Tnw4Q&feature=emb_logo

I. Pre-lesson

Key Points

Properties of common logarithms

For any $M, N > 0$,

(a) $\log 10 = 1$

(b) $\log 1 = 0$

(c) $\log(MN) = \log M + \log N$

(d) $\log\left(\frac{M}{N}\right) = \log M - \log N$

(e) $\log M^n = n \log M$, where n is a real number

1. Determine whether each of the following is correct. Put a '✓' in the box if it is correct; otherwise, put a '×' in the box.

(a) $\log(7-2) = \frac{\log 7}{\log 2}$ ☐ **(b)** $\log 4 + \log 8 = \log(4 \times 8)$ ☐

(c) $\log 10 - \log 9 = \log\left(\frac{9}{10}\right)$ ☐ **(d)** $\log(5+3) = \log 5 \times \log 3$ ☐

(e) $(\log 5)^2 = 2\log 5$ ☐ **(f)** $\log\left(\frac{6}{3}\right) = \frac{\log 6}{\log 3}$ ☐

II. Teaching example and classwork

Level 1

Find the values of the following expressions without using a calculator. **(2 – 4)**

2. **(a)** $\log 2 + \log 50$ **(b)** $\log 60 - \log 6$

3. **(a)** $\log\left(\frac{1}{4}\right) - 2\log 5$ **(b)** $3\log\left(\frac{2}{3}\right) - \log 8 + \log 27$

4. **(a)** $\dfrac{\log 27}{\log 3}$ **(b)** $\dfrac{\log 8}{\log 16}$

5. Simplify the following expressions, where $x > 0$ and $x \neq 1$.

(a) $\log(100x) + \log(\frac{1}{x})$ **(b)** $\log(5x) - \log(\frac{x}{2})$

(c) $\log(27x^3) - 3\log(3x)$ **(d)** $\dfrac{2\log x}{\log x^3 + \log x}$

(e) $\dfrac{\log x^4}{\log x^6}$ **(f)** $\dfrac{\log x^3 - \log x^2}{2\log x}$ **(g)** $2\log(5x) + \log\left(\dfrac{4}{x^3}\right) + \log x$

Level 2

6. Simplify the following expressions, where $x > 0$ and $x \neq 1$.

(a) $\dfrac{\log x^2 + \log\left(\dfrac{1}{x^3}\right)}{\log\sqrt{x}}$ **(b)** $\dfrac{\log\sqrt{x} + \log x^3}{\log x - \log\sqrt{x}}$

7. Simplify the following expressions, where $x > 0$, $y > 0$ and $x, y \neq 1$.

(a) $\log(xy) + \log\left(\frac{x}{y}\right) - 2\log x$ **(b)** $\frac{\log(x^2y^3) - 2\log(xy)}{\log\sqrt[3]{y}}$

8. Given that $\log 2 = a$, express the following in terms of a.

(a) $\log 16$ **(b)** $\log\left(\frac{1}{32}\right)$ **(c)** $\log\sqrt{8}$

(d) $\log 200$ **(e)** $\log 5$ **(f)** $\log 0.4$

9. Given that $\log 2 = x$ and $\log 3 = y$, express the following in terms of x and y.

(a) $\log 24$ **(b)** $\log 45$ **(c)** $\log\left(\frac{27}{5}\right)$

10. Given that $\log 3 = x$ and $\log 4 = y$, express the following in terms of x and y.

(a) $\log 36$ **(b)** $\log 1.2$ **(c)** $\log 18$

C. Section 8.2 Logarithms with an Arbitrary Base

I. Pre-lesson

Key Points

1. Definition of logarithms to an arbitrary base

   For $a>0$ and $a\neq 1$,

   (a) if $x=a^y$, then $y=\log_a x$;

   (b) if $y=\log_a x$, then $x=a^y$.

   Note: $\log_a x$ is undefined for $x\leq 0$.

2. Properties of logarithms to an arbitrary base

   For $a>0$ and $a\neq 1$, $b>0$ and $b\neq 1$,

   any $M, N>0$,

   (a) $\log_a a=1$

   (b) $\log_a 1=0$

   (c) $\log_a(MN)=\log_a M+\log_a N$

   (d) $\log_a\left(\frac{M}{N}\right)=\log_a M-\log_a N$

   (e) $\log_a M^n=n\log_a M$, where n is a real number

   (f) $\log_a N=\frac{\log_b N}{\log_b a}$

1. Complete the following table.

	Exponential form ($x=a^y$)	Logarithmic form ($\log_a x=y$)
(a)	$49=7^2$	
(b)		$\log_4 64=3$
(c)	$\frac{1}{125}=5^{-3}$	
(d)		$\log_3\left(\frac{1}{81}\right)=-4$

Find the values of the following logarithms without using a calculator.

2. **(a)** $\log_2 4$ **(b)** $\log_2 16$ **(c)** $\log_3 81$

II. Teaching Example and Classwork

Level 1

3. Find the values of the following logarithms without using a calculator.

(a) $\log_3\left(\frac{1}{3}\right)$ **(b)** $\log_4\left(\frac{1}{16}\right)$ **(c)** $\log_2\sqrt{8}$

4. Find the values of the following logarithms correct to 3 significant figures.

(a) $\log_2 96$ **(b)** $\log_{1.3}\left(\frac{1}{2}\right)$

Find the values of the following expressions without using a calculator. **(5 – 9)**

5. **(a)** $\log_3 18+\log_3\left(\frac{1}{2}\right)$ **(b)** $\log_2 160-\log_2 5$

6. **(a)** $\log_4 8+5\log_4 2$ **(b)** $2\log_5 15-\log_5 9$

7. **(a)** $\dfrac{\log_7 3}{\log_7 9}$ **(b)** $\dfrac{\log_5 \sqrt{8}}{\log_5 4}$

8. **(a)** $4\log_3 2 - 2\log_3 6 - \log_3 4$ **(b)** $\log_{12}\left(\dfrac{2}{3}\right) + 2\log_{12} 6 - 2\log_{12}\sqrt{2}$

9. **(a)** $\log_{16} 64$ **(b)** $\log_9\left(\dfrac{1}{27}\right)$

Level 2

10. Simplify the following expressions, where $x > 0$ and $x \neq 1$.

(a) $3\log_3 x^2 + \log_3\left(\frac{1}{x^6}\right)$

(b) $\log_4\left(\frac{x^2}{2}\right) - \log_4(8x^2)$

(c) $\frac{3\log_2 \sqrt{x} - \log_2 x}{\log_2 x^3}$

(d) $\frac{\log_3(32x^3) - \log_3(2x)}{\log_3(4x)}$

(e) $\log_5 \sqrt{x} - \frac{5}{2}\log_5 x + 2\log_5\left(\frac{x}{5}\right)$

(f) $\frac{\log_{16} x^8}{\log_4 x^5}$

(g) $\frac{\log_2 (3x)}{\log_2 27 + 3\log_2 x}$

11. Given that $\log_3 2 = a$ and $\log_3 5 = b$, express the following in terms of a and/or b.

(a) $\log_3 10$ **(b)** $\log_3\left(\frac{2}{25}\right)$ **(c)** $\log_3 15$

12. Given that $\log_2 3 = a$ and $\log_2 5 = b$, express the following in terms of a and b.

(a) $\log_2 0.6$ **(b)** $\log_2\left(\frac{25}{9}\right)$ **(c)** $\log_2 30$

13. Given that $\log_3 2 = x$ and $\log_3 5 = y$, express the following in terms of x and y.

(a) $\log_3 \sqrt{60}$ **(b)** $\log_3\left(\frac{9}{10}\right)$ **(c)** $\log_5 2$

D. Logarithmic Functions and Their Graphs

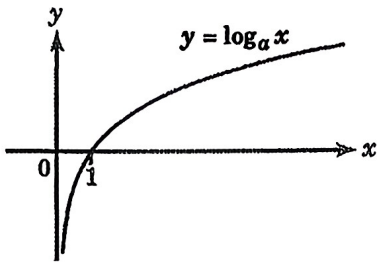
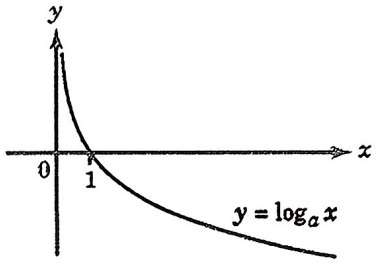
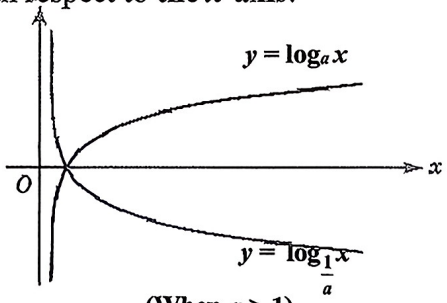
Graphs of Logarithmic Functions

(a) Let a be a constant, $a > 0$ and $a \neq 1$.

The function $y = \log_a x$ or $f(x) = \log_a x$ is called a **logarithmic function** with base a.

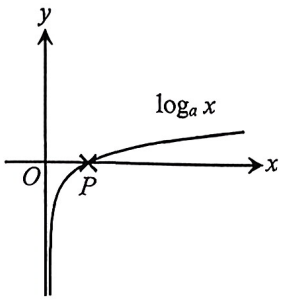
Note: When $x \leq 0$, $\log_a x$ is undefined. Hence, the **domains** of logarithmic functions are all positive real numbers.

(b) The following summarizes the features of the logarithmic functions $y = \log_a x$ and their graphs.

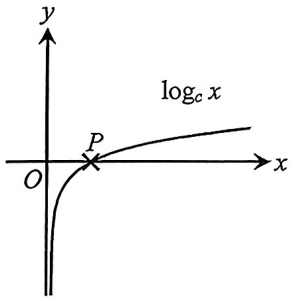
Range of a	$a > 1$	$0 < a < 1$
Graph of $y = \log_a x$		
Common features	**1.** The x-intercept of the graph is 1. ($\log_a 1 = 0$) **2.** The graph does not cut the y-axis. ($y = \log_a x$ is not defined when $x = 0$.) It lies on the right of the y-axis. (The domain is all positive real numbers.) **3.** The graphs of $y = \log_{\frac{1}{a}} x$ and $y = \log_a x$ are the **reflection images** of each other with respect to **the x-axis**:  **(When $a > 1$)**	
Different features	**1.** When $0 < x < 1$, $y < 0$. When $x > 1$, $y > 0$. **2.** The graph **slopes upwards** from left to right.	**1.** When $0 < x < 1$, $y > 0$. When $x > 1$, $y < 0$. **2.** The graph **slopes downwards** from left to right.

Level 1

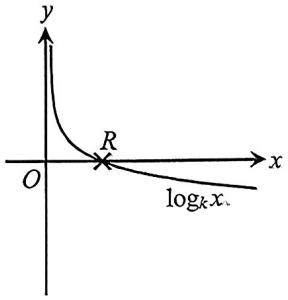
1. Refer to the figure on the right.
   (a) Write down the coordinates of P.
   (b) Write down the range of values of a.
   (c) If the graph of $y = \log_a x$ is the reflection image of the graph of $y = \log_{0.2} x$ with respect to the x-axis, find the value of a.



2. Refer to the figure on the right.
   (a) Write down the coordinates of P.
   (b) Write down the range of values of c.
   (c) If the graph of $y = \log_c x$ is the reflection image of the graph of $y = \log_{\frac{2}{3}} x$ with respect to the x-axis, find the value of c.

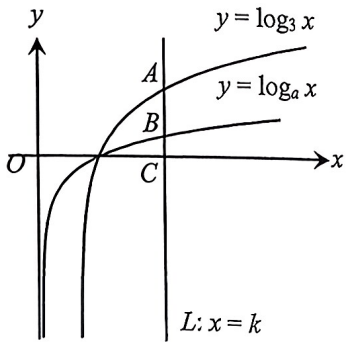


3. Refer to the figure on the right.
   (a) Write down the coordinates of R.
   (b) Write down the range of values of k.
   (c) If the graph of $y = \log_k x$ is the reflection image of the graph of $y = \log_4 x$ with respect to the x-axis, find the value of k.



Level 2

4. The figure shows the graphs of $y = \log_3 x$ and $y = \log_a x$, where a is a positive constant. A vertical line L: $x = k$ cuts the graph of $y = \log_3 x$, the graph of $y = \log_a x$ and the x-axis at A, B and C respectively.



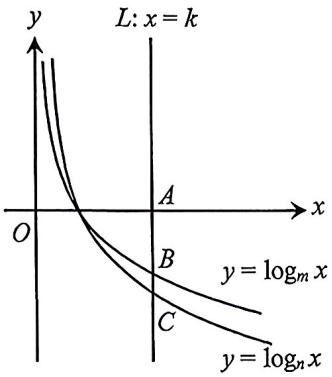
(a) Determine whether each of the following is true. Explain your answers.

(i) $ak > 1$

(ii) $a < 3$

(b) Express $\dfrac{BC}{AC}$ in terms of a.

5. The figure shows the graphs of $y = \log_m x$ and $y = \log_n x$, where m and n are positive constants. A vertical line L: $x = k$ cuts the x-axis, the graph of $y = \log_m x$ and the graph of $y = \log_n x$ at A, B and C respectively.



(a) Determine whether each of the following is true. Explain your answers.

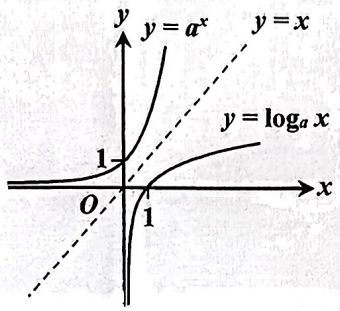
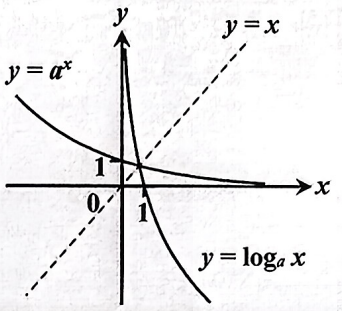
(i) $mn > 1$

(ii) $m - n > 0$

(b) Show that $\dfrac{AB}{AC} = \log_m n$.

Relationship between the Graphs of Exponential and Logarithmic Functions

For $a > 0$ and $a \neq 1$, the graphs of $y = a^x$ and $y = \log_a x$ are the reflection images of each other **with respect to the line $y = x$.**

Range of a	$a > 1$	$0 < a < 1$
Graphs of $y = a^x$ and $y = \log_a x$		

Level 1

6. In each of the following, the graph of the function is reflected with respect to the line $y = x$. Write down the corresponding function of each graph obtained.

   (a) $y = 2.5^x$ **(b)** $y = \log_{0.7} x$

7. In each of the following, the graph of the function is reflected with respect to the line $y = x$. Write down the corresponding function of each graph obtained.

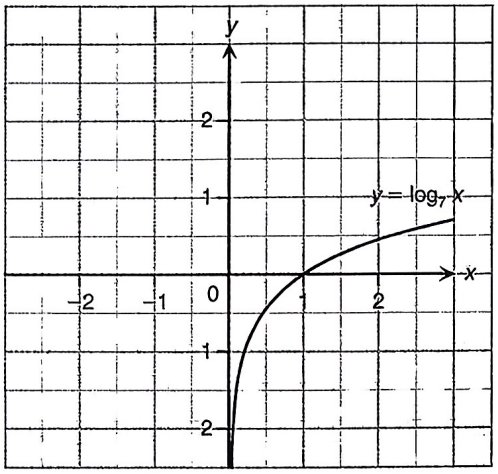
   (a) $y = 6^x$ **(b)** $y = \left(\frac{3}{8}\right)^x$

   (c) $y = \log_5 x$ **(d)** $y = \log_{\frac{1}{\sqrt{3}}} x$

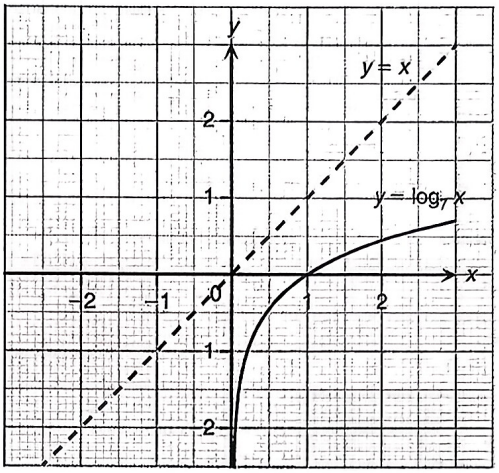
Level 2

In each of the following, according to the given graph of logarithmic functions, sketch the required graph on the same graph paper. **(8 – 9)**

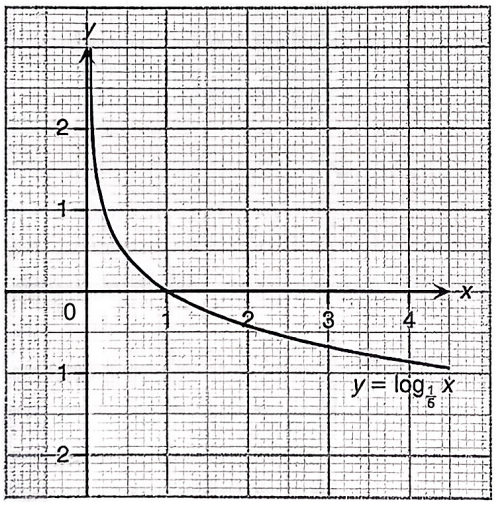
8. **(a)** graph of $y = \log_{\frac{1}{7}} x$



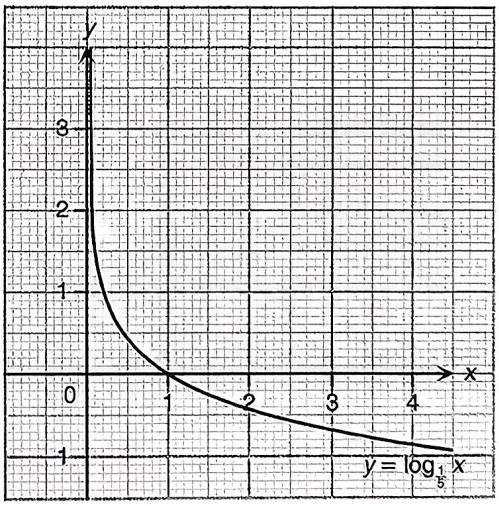
(b) graph of $y = 7^x$



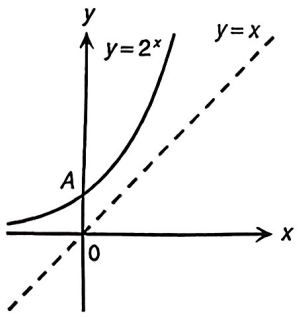
9. **(a)** graph of $y = \log_5 x$



(b) graph of $y = \left(\frac{1}{5}\right)^x$



10. The figure shows the graphs of $y = 2^x$ and $y = x$. The graph of $y = 2^x$ is reflected about the line $y = x$ to obtain the graph of $y = f(x)$.



(a) It is given that the graph of $y = 2^x$ and $y = f(x)$ cuts the y-axis at A and B respectively.

(i) Write down the coordinates of A and B.

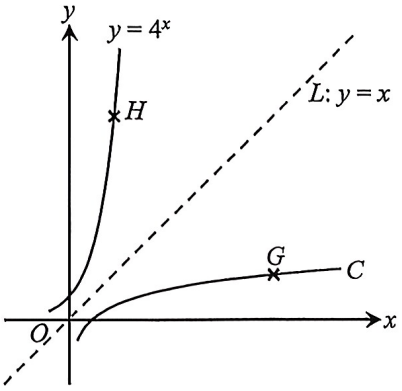
(ii) Mark B on the given graph.

(b) **(i)** Write down the algebraic representation of the function $f(x)$.

(ii) Sketch the graph of $y = f(x)$ on the given graph.

Level 3

11. In the figure, C is the reflection image of the graph of $y = 4^x$ with respect to the line $L: y = x$.



(a) Write down the corresponding function for C.

(b) It is given that the y-coordinate of the point G is 3, and the point H is the reflection image of G with respect to L.

(i) Find the coordinates of H.

(ii) Is GH perpendicular to L? Explain your answer.

When a point (a, b) is reflected with respect to the line $y = x$, the image is (b, a).

E. Section 8.4 Exponential Equations and Logarithmic Equations: Exponential Equations

Solve the following exponential equations. **(1 – 14)**

(Give your answers correct to 3 significant figures.)

Level 1

1. $5^{2x} = 5^{1-4x}$

2. $6^{x+1} = 6^{2x-1}$

3. $7^{x-1} = \frac{1}{49}$

4. $9^x = 27$

5. $4^x = 7$

6. $3^x = 10$

7. $5^{2x-3} = 9$

8. $2^{x+1} = 0.3$

Level 2

9. $5^x = 6^{x-1}$

10. $5^x = 2^{x+1}$

11. $3^x = 4^{x-1}$

12. $8^{x+1} = 7^{2-x}$

13. $7^x = 10(2^x)$

14. $7^{x+1} + 7^x = 15$

15. $6^{x+2} = 6^{x+1} + 30$

16. $5^{x-1} - 5^x + 100 = 0$

F. Section 8.4 Exponential Equations and Logarithmic Equations: Logarithmic Equations Level 1

Solve the following logarithmic equations. **(1 – 10)**

1. $\log 5 + \log(x+2) = 2$

2. $\log(4x-3) - \log 5 = -1$

3. $\frac{1}{2}\log(8x-4) = 1$

4. $\log 8 + \log 2 = \log(x+2)$

5. $\log(7x+11) = 2\log 5$

6. $\log(5x-1) - 3\log 4 = 0$

7. $\log(3x-1) - \log(x-1) = \log 5$

8. $\log(2x-7) - \log 3 = \log(x-3)$

9. $\log(3x) + 4\log 2 = \log(8x + 5)$

10. $\log(x + 7) - \log(4x + 3) = \log 4 - \log 6$

Level 2

Solve the following logarithmic equations. **(11 – 14)**

11. **(a)** $\log_2(x+1) = 4$

(b) $\log_3(2x-3) = -1$

12. **(a)** $\log_2(x-1) - \log_2 6 = -2$

(b) $\log_4(5x+2) + \log_4 3 = 2$

13. **(a)** $\log_9(3x+1) = \log_3 4$ **(b)** $\log_2(x-1) = \log_4 9$

14. **(a)** $\log_7(4x-3) = \log_7(x-3)+1$ **(b)** $\log_{49}(5x+1) - \log_7 6 = 0$

G. Section 8.2 Application of Logarithms: Basic Applications

Level 1

1. The number of bacteria (B) in a candy after n hours of exposure is given by $B = 28\,000\log(3n+1)$.

(a) Find the number of bacteria after 6 hours of exposure, correct to the nearest thousand.

(b) Given that the number of bacteria after k hours of exposure is 56 000, find the value of k.

Level 2

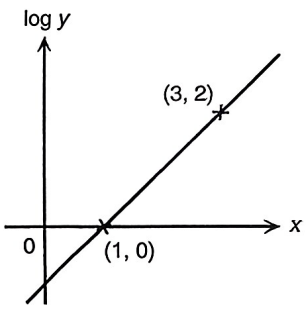
2. The monthly rent of a car park ($\$R$) increases at a constant rate of 8% per year. At least how many years will it take for the monthly rent to exceed twice its original rent?

3. Mrs Wong deposits a sum of money ($\$P$) in a bank at an interest rate of 6% p.a. compounded half-yearly. Find the least number of years for the amount to exceed three times the original amount of money.

H. Section 8.2 Application of Logarithms: Logarithmic Transformation

Level 1

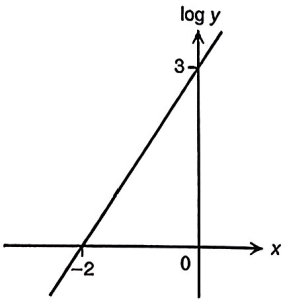
1. The figure shows the graph of $\log y$ against x. The graph passes through (1, 0) and (3, 2). It is given that $y = ab^x$, where a and b are constants.



(a) Express $\log y$ in terms of a, b and x.

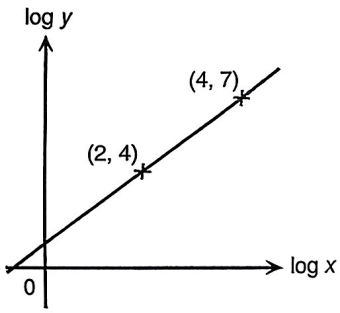
(b) Find the values of a and b.

2. The figure shows the graph of $\log y$ against x for a relationship $y = ka^x$, where k and a are positive constants. Find the values of k and a.



Level 2

3. The figure shows the graph of $\log y$ against $\log x$. The graph passes through (2, 4) and (4, 7). It is given that $\log y = a\log x + b$, where a and b are constants.

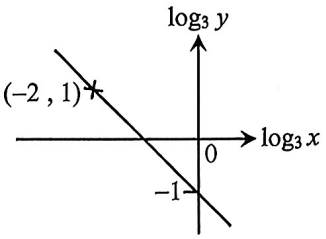


(a) Find the values of a and b.

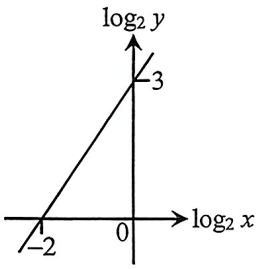
(b) Express y in terms of x.

(c) Find the percentage change in the value of y when x is increased by 50%.

4. The graph in the figure shows the linear relation between $\log_3 x$ and $\log_3 y$. The graph passes through $(-2\ ,\ 1)$ and the intercept on the vertical axis of the graph is -1. It is given that $y = bx^n$, where b and n are constants. Find the values of b and n.



5. The graph in the figure shows the linear relation between $\log_2 x$ and $\log_2 y$. The intercepts on the vertical axis and on the horizontal axis of the graph are 3 and -2 respectively. It is given that $y = ax^b$, where a and b are constants. Find the values of a and b.



6. It is given that $\log_4 y$ is a linear function of $\log_4 x$. The graph of the linear function passes through $(1 , 2)$ and $(-5 , 8)$. Express y in terms of x.

7. It is given that $\log_5 y$ is a linear function of $\log_5 x$. The graph of the linear function passes through $(-2 , 7)$ and the slope of the graph is -3. Express y in terms of x.

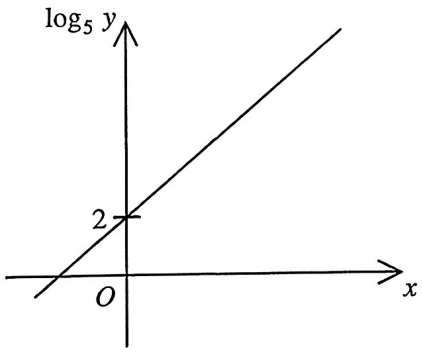
Level 3

8. It is given that $\log_9 y$ is a linear function of $\log_3 x$. The intercepts on the horizontal axis and on the vertical axis of the graph of the linear function are 2 and 1 respectively. Express y in terms of x.

Past Paper

9. The graph in the figure shows the linear relation between x and $\log_5 y$. If $y = ab^x$, then $a =$

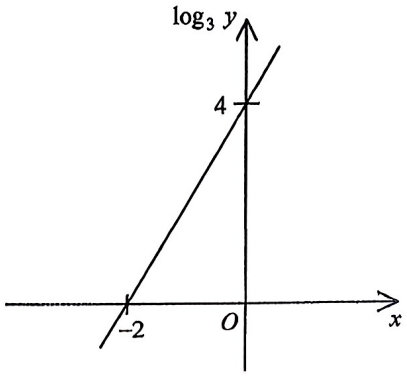
A. 1
B. 2
C. 5
D. 25



[HKDSE SP #32]

10. The graph in the figure shows the linear relation between x and $\log_3 y$. If $y = mn^x$, then $n =$

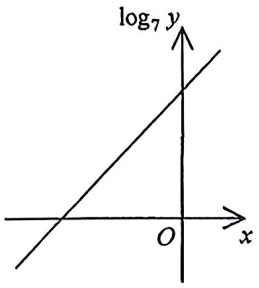
A. $\frac{1}{81}$
B. $\frac{1}{9}$
C. 9
D. 81



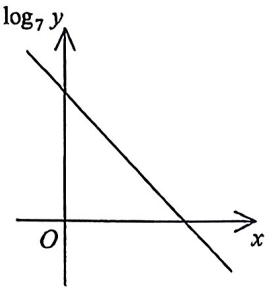
[HKDSE 12 #32]

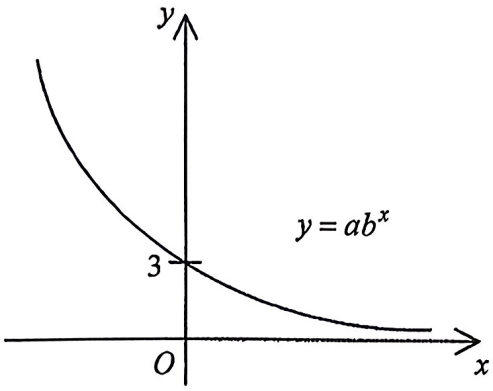
11. The figure above shows the graph of $y = ab^x$, where a and b are constants. Which of the following graphs may represent the relation between x and $\log_7 y$?

A.

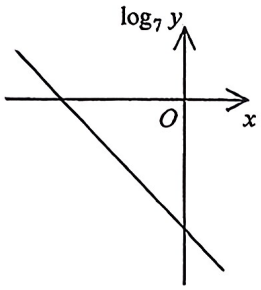


B.

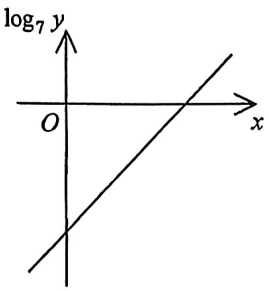




C.



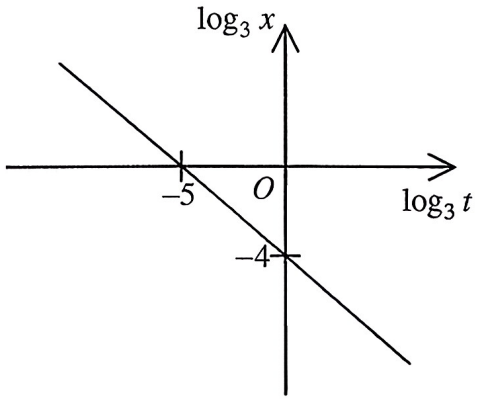
D.



[HKDSE 13 #32]

12. The graph in the figure shows the linear relation between $\log_3 t$ and $\log_3 x$. If $x = kt^a$, then $k =$

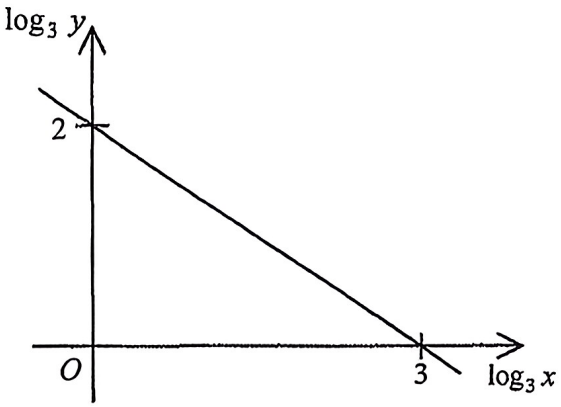
A. $\frac{1}{81}$
B. 81 .
C. $\frac{-4}{5}$
D. $\frac{-5}{4}$



[HKDSE PP #37]

13. The graph in the figure shows the linear relation between $\log_3 x$ and $\log_3 y$. Which of the following must be true?

A. $x^2y^3 = 729$
B. $x^3y^2 = 729$
C. $x^2 + y^3 = 729$
D. $x^3 + y^2 = 729$



[HKDSE 15 #32]

I. Section 8.5 Application of Logarithms: Sound Intensity Level & Richter Scale

You may refer to the following formulas to answer all the questions in this worksheet.

(a) The sound intensity level β (in dB) of a sound is defined as:

$$\beta = 10\log\left(\frac{I}{I_0}\right),$$

where I is the sound intensity (in W/m^2) measured, and $I_0 = 10^{-12}$ W/m^2.

(b) The magnitude (M) of an earthquake measured on the Richter scale is given by:

$$M = \frac{2}{3}\log E + K,$$

where E is the energy (in J) released in the earthquake, and K is a constant.
(Unless otherwise specified, give your answers in scientific notation and correct to 3 significant figures if necessary.)

Level 1

1. Given that the sound intensity of a music concert is 6.3×10^{-3} W/m^2, find its sound intensity level.

2. Given that the sound intensity level of the noise from a busy street is 94 dB, find its sound intensity.

Level 2

3. The sound intensity of loudspeaker A is 3 times that of loudspeaker B. What is the difference between their corresponding sound intensity levels?

4. The magnitudes of the earthquakes occurred in city X and city Y were measured 4.5 and 3.2 respectively on the Richter scale. How many times of the energy released in the earthquake in city Y was the energy released in the earthquake in city X?