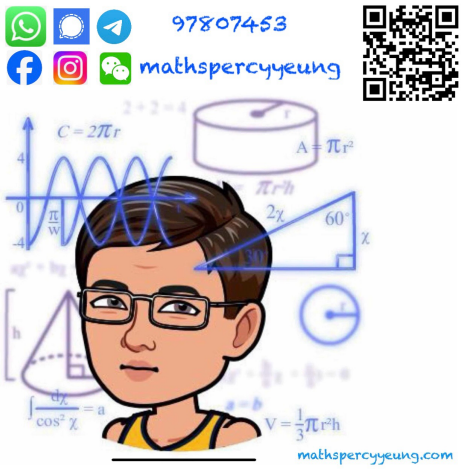
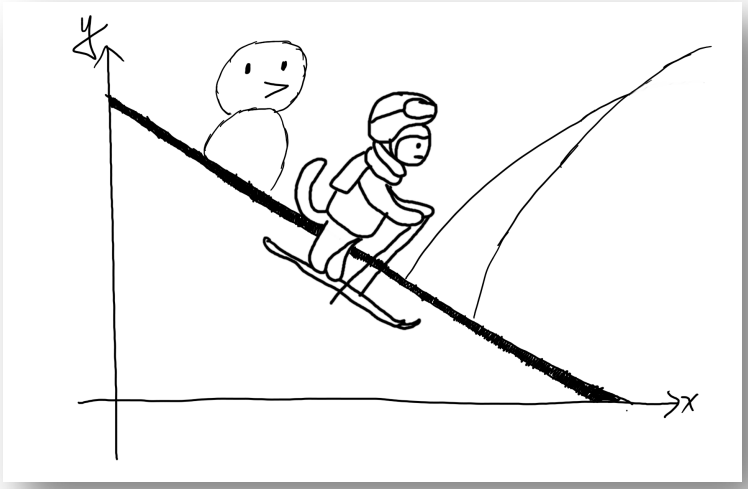
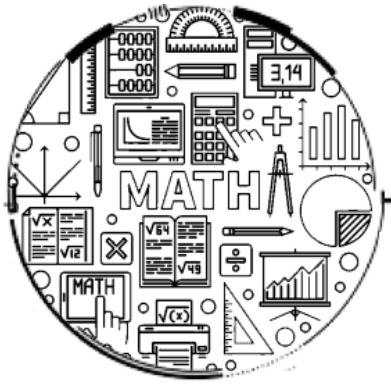
S3 Mathematics Notes







Coordinate Geometry of Straight Lines

Content:

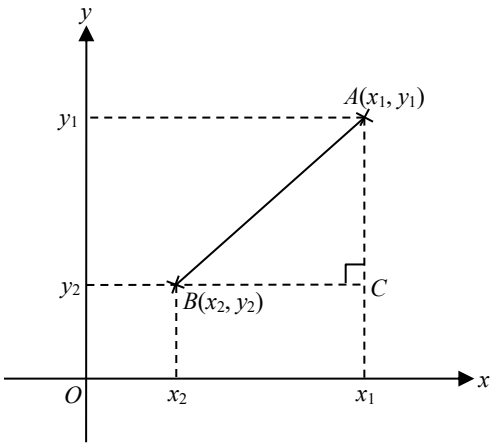
- Distance Formula
- Slopes of Straight Lines
- Slopes of Parallel Lines and Perpendicular Lines
- Point of Division
- Analytic Approach

A Distance Formula

Key Points: Distance Formula

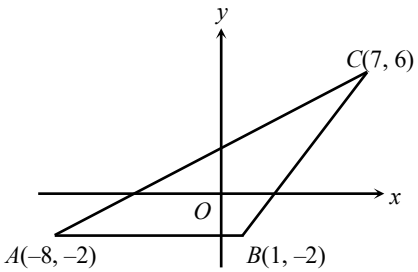
The distance between any two points $A(x_1, y_1)$ and $B(x_2, y_2)$ on a rectangular coordinate plane is given by

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \ .$$



Practice Exercise

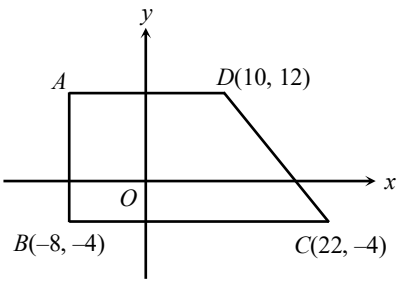
1. In the figure, $A(-8, -2)$, $B(1, -2)$ and $C(7, 6)$ are the vertices of $\triangle ABC$. Find the perimeter of $\triangle ABC$.



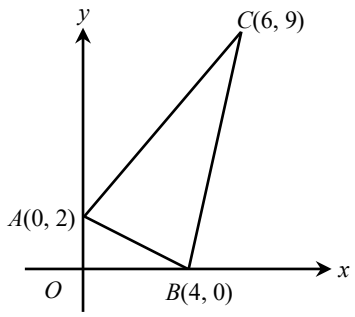
2. In the figure, A, $B(-8, -4)$, $C(22, -4)$ and $D(10, 12)$ are the vertices of the trapezium $ABCD$, where $AD \parallel BC$ and $AB \perp BC$.

(a) Find the coordinates of A.

(b) Find the perimeter of $ABCD$.

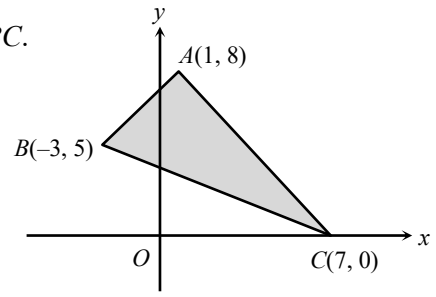


3. In the figure, $A(0, 2)$, $B(4, 0)$ and $C(6, 9)$ are the vertices of $\triangle ABC$. Show that $\triangle ABC$ is an isosceles triangle.



4. $P(-10, 4)$, $Q(5, -4)$ and $R(-10, -12)$ are the vertices of $\triangle PQR$. Is $\triangle PQR$ an equilateral triangle? Explain your answer.

5. In the figure, $A(1, 8)$, $B(-3, 5)$ and $C(7, 0)$ are the vertices of $\triangle ABC$.



(a) Show that $\triangle ABC$ is a right-angled triangle.

(b) Find the area of $\triangle ABC$.

6. The coordinates of A, B and C are $(3, 0)$, $(-2, k)$ and $(8, k)$ respectively, where $k \neq 0$.

(a) Prove that $\triangle ABC$ is an isosceles triangle.

(b) If $AB \perp AC$, find all possible values of k.

Example 1 (TB P.9.6)

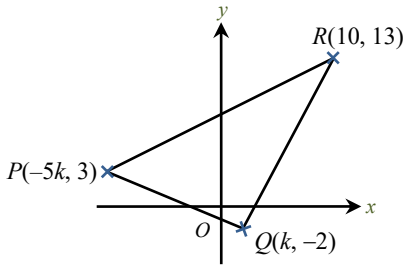
The coordinates of points P and Q are $(2k, 6)$ and $(-k, -3)$ respectively, where k is a constant. If $PQ = 15$ units, find all possible values of k.

Practice Exercise

7. In the figure, $P(-5k, 3)$, $Q(k, -2)$ and $R(10, 13)$ are the vertices of $\triangle PQR$, where P lies in quadrant II. It is given that $PQ = 13$ units.

(a) Find the value of k.

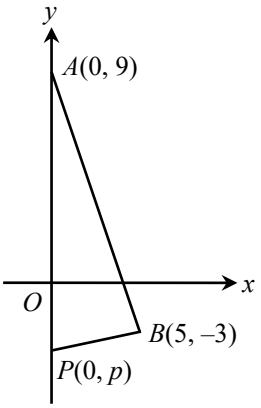
(b) Wincy claims that the perimeter of $\triangle PQR$ exceeds 50 units. Do you agree? Explain your answer.



8. The points $I(-3k, k)$, $J(4k, 2k)$, K and L are the vertices of a square on a rectangular coordinate plane. If the area of square $IJKL$ is 3200 sq. units, find all possible values of k.

Example 2 (TB P.9.6)

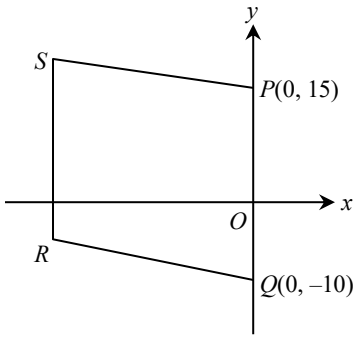
In the figure, $A(0, 9)$ and $B(5, -3)$ are two points on a coordinate plane. $P(0, p)$ is a point on the negative y-axis such that $AP = AB$. Find the value of p.



Practice Exercise

9. $P(-3, -1)$, $Q(0, 3)$ and $R(k, 3)$ are the vertices of $\triangle PQR$. If $k > 0$ and $PQ = QR$, find the value of k.

10. In the figure, *PQRS* is a rhombus on a rectangular coordinate plane, where the coordinates of points *P* and *Q* are (0, 15) and (0, –10) respectively. It is given that the *y*-coordinate of *R* is –3. Find the area of rhombus *PQRS*.



11. *PQRS* is a parallelogram on a rectangular coordinate plane, where the coordinates of *Q* and *S* are (18, 0) and (0, 12) respectively. It is given that *P* lies on the positive *x*-axis such that *PQ* = *PS*.

(a) Find the coordinates of *P* and *R*.

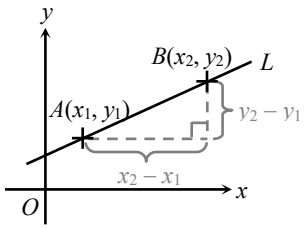
(b) Find the area of parallelogram *PQRS*.

B Slopes of Straight Lines

Key Points: Slope Formula

Consider a straight line L passing through the points $A(x_1, y_1)$ and $B(x_2, y_2)$, where $x_1 \neq x_2$. Then the slope m of L is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$



Example 3 (TB P.9.15)

Given that the slope of the line passing through $A(-3\ ,\ 2)$ and $B(k\ ,\ 4k)$ is 2, find the value of k.

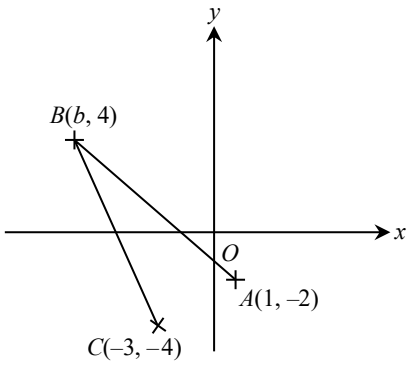
Practice Exercise

12. Given that the slope of the line passing through $A(1 - n\ ,\ 6)$ and $B(n\ ,\ 3)$ is -1, find the value of n.

13. The coordinates of points A, B and C are (1, –2), (b, 4)and (–3, –4) respectively, where b is a constant. If the slope of AB is $-\frac{6}{7}$, find

(a) the value of b,

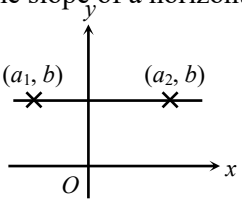
(b) the slope of BC.



14. The coordinates of points A, B and C are (1, 2), (–3, b) and (–9, 8) respectively, where b is a constant. If the slope of BC is twice that of AB, find the value of b.

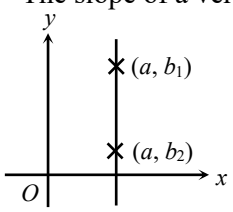
Mathematical Skills: Slopes of Horizontal Lines and Vertical Lines

The slope of a horizontal line is 0.



$$\text{Slope} = \frac{b-b}{a_2-a_1} = 0$$

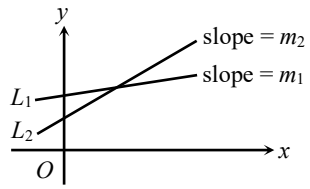
The slope of a vertical line is undefined.



$$\text{Slope} = \frac{b_2-b_1}{a-a} = \frac{b_2-b_1}{0} \text{ (undefined)}$$

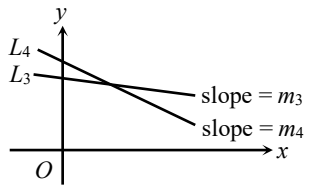
Key Points: Properties of Slopes of Straight Lines

(a) Straight lines **sloping upwards** from left to right



- The slope is **positive**. ($m_1 > 0$ and $m_2 > 0$)
- If $m_2 > m_1$, L_2 is steeper than L_1.

(b) Straight lines **sloping downwards** from left to right



- The slope is **negative**. ($m_3 < 0$ and $m_4 < 0$)
- If $m_3 > m_4$, L_4 is steeper than L_3.

Let's Try

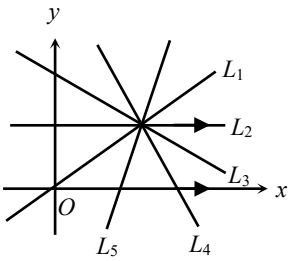
1. In the figure, L_1, L_2, L_3, L_4 and L_5 are straight lines and their slopes are m_1, m_2, m_3, m_4 and m_5 respectively.

(a) Write down the straight line(s) with

(i) zero slope; _____

(ii) positive slope; _____

(iii) negative slope. _____



(b) Arrange m_1, m_2, m_3, m_4 and m_5 in ascending order.

_____ < _____ < _____ < _____ < _____

In each of the following, determine which straight line, L_1 or L_2, is steeper. Explain your answer. **(2 – 3)**

2. Slope of $L_1 = \frac{1}{3}$, slope of $L_2 = 1$

∵ The slopes of L_1 and L_2 are both (positive / negative) and

slope of (L_1 / L_2) > slope of (L_1 / L_2).

∴ (L_1 / L_2) is steeper than (L_1 / L_2).

3. Slope of $L_1 = -\frac{1}{2}$, slope of $L_2 = -2$

Practice Exercise

15. L_1 passes through the origin O and the point $(6, -4)$, while L_2 passes through the points $(-5, 3)$ and $(7, -1)$. Which straight line, L_1 or L_2, is steeper? Explain your answer.

16. The coordinates of points A and C are $(10, 6)$ and $(-8, -5)$ respectively. B is a point lying on the y-axis such that the slope of AB is $\frac{2}{5}$.

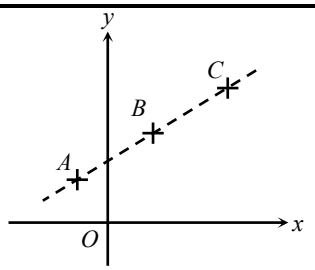
(a) Find the coordinates of B.

(b) Find the slope of BC.

(c) Which straight line, AB or BC, is steeper? Explain your answer.

Key Points: Collinear

If slope of AB = slope of BC,
(common point)
then A, B and C are collinear.
(i.e. The three points lie on the same straight line.)



Example 4 (TB P.9.16)

In each of the following, determine whether the three given points are **collinear**.

(a) $A(-4, -6)$, $B(0, -2)$, $C(3, 1)$

(b) $P(-3, 6)$, $O(0, 0)$, $Q(4, -12)$

Practice Exercise

17. In each of the following, determine whether the three given points are collinear.

(a) $I(-2, -3)$, $J(2, -1)$, $K(6, 1)$

(b) $P(-7, -9)$, $Q(-2, -1)$, $R(8, 11)$

18. The coordinates of points A, B, C and D are $(0, 7)$, $(-4, 1)$, $(-8, -5)$ and $(d, d + 2)$ respectively, where d is a constant.

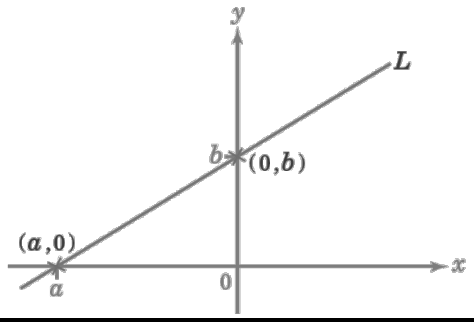
(a) Prove that A, B and C are collinear.

(b) If B, C and D are collinear, find d.

Key Points: Intercepts of Straight Lines

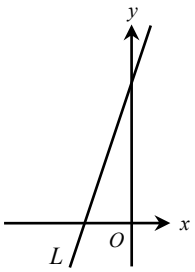
In the figure, the straight line L cuts the x-axis and the y-axis at $(a\ ,\ 0)$ and $(0\ ,\ b)$ respectively.

a and b are called the ***x*-intercept** and the ***y*-intercept** of L respectively.



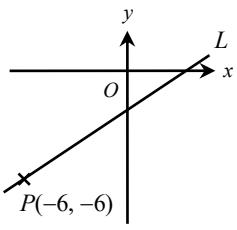
Example 5 (TB P.9.17)

In the figure, L is a straight line with slope 3. The y-intercept of L is 12. Find the x-intercept of L.



Practice Exercise

19. In the figure, a straight line L with slope $\frac{2}{3}$ passes through the point $P(-6, -6)$. Find the x-intercept and the y-intercept of L.



20. If a straight line passes through points $P(-3, 8)$ and $Q(6, -4)$, find the x-intercept and the y-intercept of the straight line.

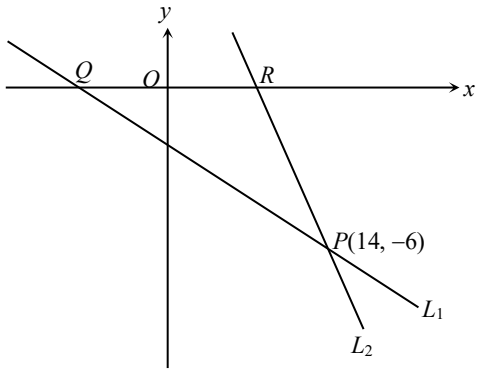
21. The x-intercept and the y-intercept of a straight line are -2 and 4 respectively. If the straight line passes through points $A(a, -2)$ and $B(3, b)$, where a and b are constants, find the values of a and b.

22. In the figure, straight lines L_1 and L_2 intersect at $P(14, -6)$. L_1 and L_2 cut the x-axis at Q and R respectively. It is given that the slope of L_2 is $-\frac{3}{2}$ and the area of $\triangle PQR$ is 54 sq. units.

(a) Find the coordinates of R.

(b) Find the length of QR.

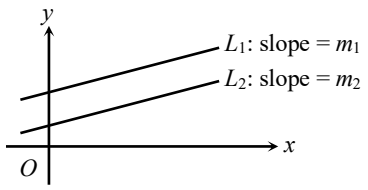
(c) Jayden claims that the y-intercept of L_1 is less than -2. Do you agree? Explain your answer.



C Slopes of Parallel Lines and Perpendicular Lines

Key Points: Properties of Parallel Lines

Consider two non-vertical straight lines L_1 and L_2 with slopes m_1 and m_2 respectively.

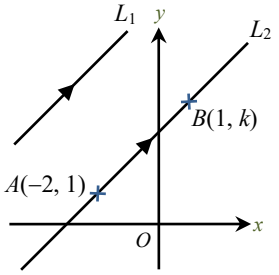


(a) If $L_1 \parallel L_2$, then $m_1 = m_2$.

(b) If $m_1 = m_2$, then $L_1 \parallel L_2$.

Example 6 (TB P.9.25)

In the figure, L_1 is a straight line with slope 1 and L_2 is a straight line passing through the points $A(-2, 1)$ and $B(1, k)$. It is given that $L_1 \parallel L_2$. Find the value of k.



Practice Exercise

23. The coordinates of points A, B, C and D are $(k, 8)$, $(9, 12)$, $(-2k, -3)$ and $(6, 5)$ respectively, where k is a constant. If $AB \parallel CD$, find the value of k.

24. The coordinates of points P, Q and R are $(-14, -15)$, $(-10, -7)$ and $(6, 9)$ respectively. S is a point lying on the y-axis such that $PQ \parallel SR$. Find the coordinates of S.

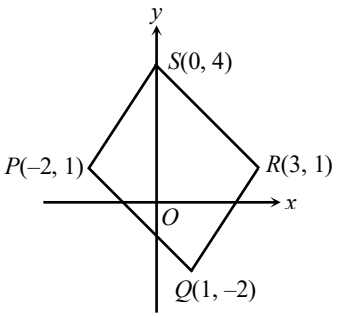
25. The coordinates of points A, B, C and D are $(-4, 4k)$, $(2, 12)$, $(6, k-2)$ and $(10, -4)$ respectively, where k is a constant. It is given that $AB \parallel CD$.

(a) Find the value of k.

(b) By considering the x-intercepts and the y-intercepts of the straight lines AB and CD, determine whether A, B, C and D are collinear.

Example 7 (TB P.9.25)

In the figure, $P(-2, 1)$, $Q(1, -2)$, $R(3, 1)$ and $S(0, 4)$ are the vertices of a quadrilateral $PQRS$. Show that $PQRS$ is a parallelogram.



Practice Exercise

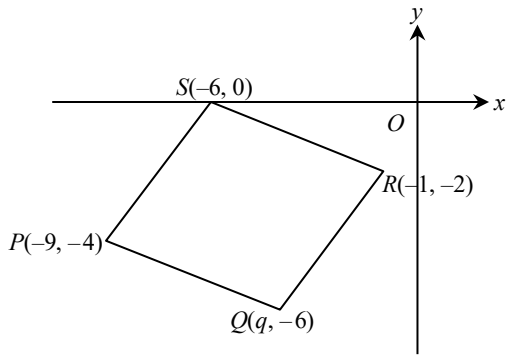
26. The coordinates of points A, B, C and D are $(-7, 2)$, $(-8, -3)$, $(-4, -1)$ and $(-3, 4)$ respectively. Prove that $ABCD$ is a parallelogram.

27. In the figure, the coordinates of points P, Q, R and S are $(-9, -4)$, $(q, -6)$, $(-1, -2)$ and $(-6, 0)$ respectively.

It is given that the slopes of PQ is $-\frac{2}{5}$.

(a) Find q.

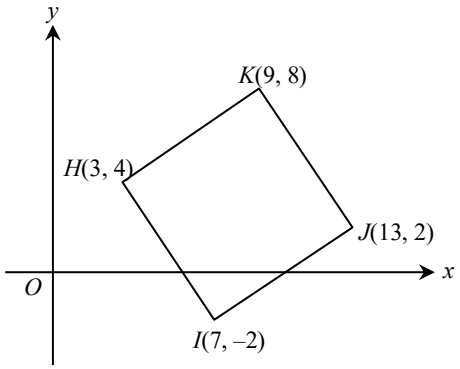
(b) Prove that $PQRS$ is a parallelogram.



28. The coordinates of points H, I, J and K are $(3, 4)$, $(7, -2)$, $(13, 2)$ and $(9, 8)$ respectively.

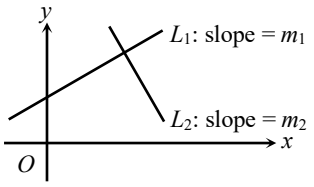
(a) Prove that $HIJK$ is a parallelogram.

(b) Is $HIJK$ a rhombus? Explain your answer.



Key Points: Properties of Perpendicular Lines

Consider two straight lines L_1 and L_2 with slopes m_1 and m_2 respectively, where $m_1, m_2 \neq 0$.



(a) If $L_1 \perp L_2$, then $m_1 \times m_2 = -1$.

(b) If $m_1 \times m_2 = -1$, then $L_1 \perp L_2$.

Practice Exercise

29. It is given that the coordinates of A and B are $(-2, 5)$ and $(7, -7)$ respectively. A is rotated clockwise about the origin through 90° to A'. B is reflected about the y-axis to B'.

(a) Find the coordinates of A' and B'.

(b) Prove that AB is perpendicular to $A'B'$.

30. The coordinates of the points P and Q are $(4, -2)$ and $(-12, 6)$ respectively. P is rotated anticlockwise about the origin through 270° to P'. Q is translated rightwards by 15 units to Q'.

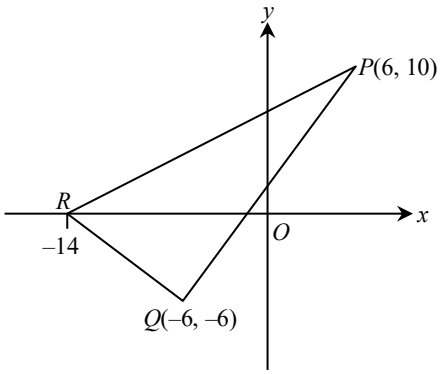
(a) Write down the coordinates of P' and Q'.

(b) Prove that PQ is perpendicular to $P'Q'$.

31. In the figure, the coordinates of P and Q are (6, 10) and (–6, –6) respectively.

(a) Prove that $\triangle PQR$ is a right-angled triangle.

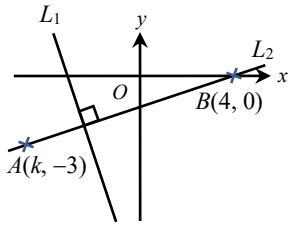
(b) Find the area of $\triangle PQR$.



32. The coordinates of points P, Q, R and S are (–6, –4), (–8, –12), (8, –16) and (10, –8) respectively. Prove that $PQRS$ is a rectangle.

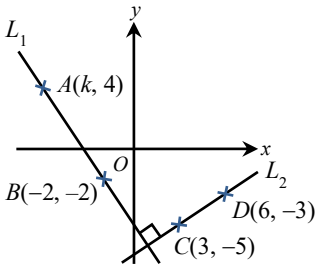
Example 8 (TB P.9.29)

In the figure, L_1 is a straight line with slope -3 and L_2 is a straight line passing through the points $A(k, -3)$ and $B(4, 0)$. It is given that $L_1 \perp L_2$. Find the value of k.



Practice Exercise

33. In the figure, L_1 is a straight line passing through the points $A(k, 4)$ and $B(-2, -2)$. L_2 is a straight line passing through the points $C(3, -5)$ and $D(6, -3)$. It is given that L_1 is perpendicular to L_2. Find the value of k.



34. $A(k, -3)$, $B(0, -8)$ and $C(10, -2)$ are the vertices of $\triangle ABC$. It is given that $\triangle ABC$ is a right-angled triangle with $\angle ABC = 90°$.

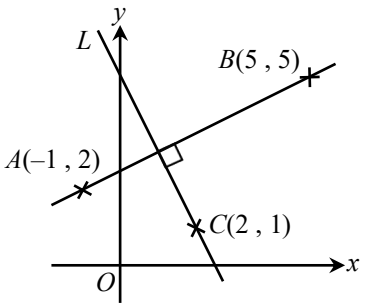
(a) Find the value of k.

(b) Find the area of $\triangle ABC$.

35. In the figure, L is perpendicular to the line passing through $A(-1\ ,\ 2)$ and $B(5\ ,\ 5)$. $C(2\ ,\ 1)$ is a point on L.

(a) Find the slope of AB.

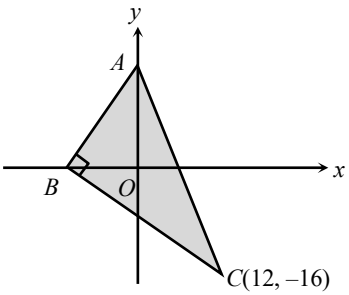
(b) Find the y-intercept of L.



36. Refer to the figure. It is given that the slope of BC is $-\frac{4}{5}$.

(a) Find the coordinates of A and B.

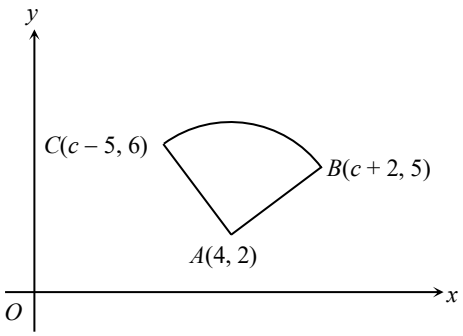
(b) Find the area of $\triangle ABC$.



37. In the figure, $A(4, 2)$ is the centre of sector ABC. The coordinates of B and C are $(c + 2, 5)$ and $(c - 5, 6)$ respectively.

(a) Find the value of c.

(b) Find the area of sector ABC.

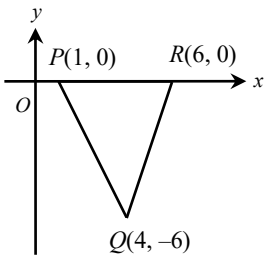


Example 9 (TB P.9.32)

In the figure, $P(1, 0)$, $Q(4, -6)$ and $R(6, 0)$ are the vertices of $\triangle PQR$.

(a) Denote the orthocentre of $\triangle PQR$ by H. Show that H lies on the vertical line passing through Q.

(b) Find the coordinates of H.



Practice Exercise

38. In the figure, the straight line L cuts the x-axis and the y-axis at $A(-16, 0)$ and $B(0, -12)$ respectively. Let C be a point lying on the positive y-axis such that the x-coordinate of the orthocentre of $\triangle ABC$ is -6. Find the y-coordinate of C.

39. The straight line L cuts the x-axis and the y-axis at points $P(-4, 0)$ and $Q(0, 10)$ respectively. Let R be a point lying on the positive x-axis such that the y-coordinate of the orthocentre of $\triangle PQR$ is 4.

(a) Find the x-coordinate of R.

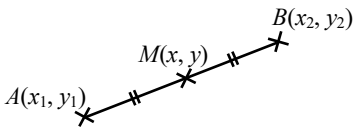
(b) Hence, or otherwise, find the area of $\triangle PQR$.

D Point of Division

Key Points: Mid-point Formula

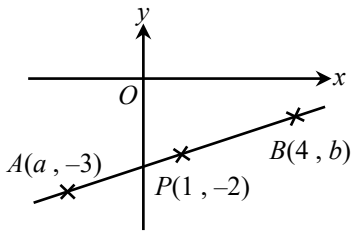
If $M(x, y)$ is the mid-point of the line segment joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$, then

$$x = \frac{x_1 + x_2}{2} \quad \text{and} \quad y = \frac{y_1 + y_2}{2}.$$



Example 10 (TB P.9.41)

In the figure, $P(1 , -2)$ is the mid-point of the line segment joining $A(a , -3)$ and $B(4 , b)$. Find the values of a and b.



Practice Exercise

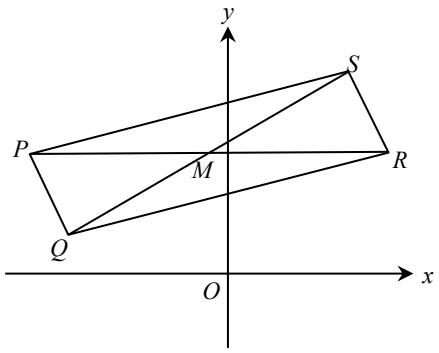
40. It is given that $M(12, 9)$ is the mid-point of the line segment joining points $P(-8, 1)$ and Q. Find the coordinates of Q.

41. The coordinates of points A and B are $(-3a, a-4)$ and $(17, b)$ respectively, where a and b are constants. $M(-8, \frac{-a+5}{2})$ is the mid-point of the line segment AB.

(a) Find the values of a and b.

(b) Find the length of AB, correct to 3 significant figures.

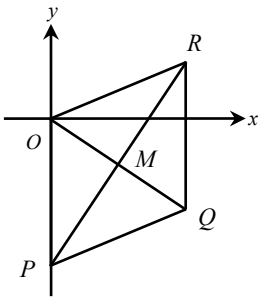
42. In the figure, $PQRS$ is a parallelogram. Its diagonals PR and QS intersect at point M, where PR is a horizontal line segment. The x-coordinate of P is -10. The coordinates of S and M are $(6, 10)$ and $(-1, 6)$ respectively. Find the coordinates of Q and R.



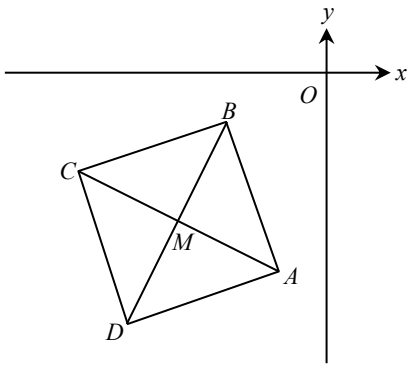
43. In the figure, $OPQR$ is a rhombus, where OP and QR are vertical line segments. OQ and PR intersect at $M(6, -4)$.

(a) Find the coordinates of Q.

(b) Find the coordinates of P and R.



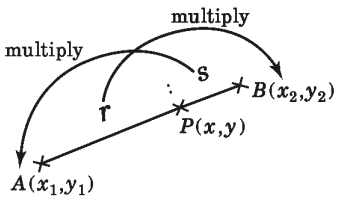
44. In the figure, $ABCD$ is a square. Its diagonals AC and BD intersect at point M. It is given that the coordinates of points C and M are $(-10, -4)$ and $(-6, -6)$ respectively and the slope of AD is $\frac{1}{3}$. Find the coordinates of A, B and D.



Key Points: Section Formula

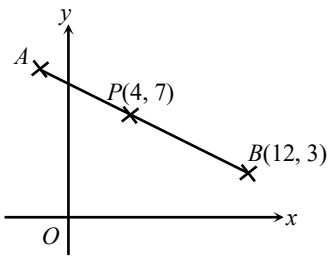
If $P(x, y)$ is a point on the line segment joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$ such that $AP : PB = r : s$, then

$$x = \frac{sx_1 + rx_2}{r + s} \quad \text{and} \quad y = \frac{sy_1 + ry_2}{r + s}.$$



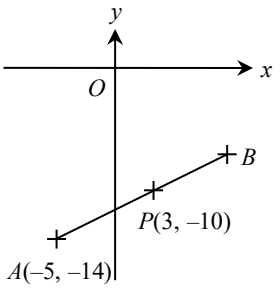
Example 11 (TB P.9.48)

In the figure, P is a point on the line segment AB such that $AP : PB = 3 : 4$. The coordinates of P and B are (4, 7) and (12, 3) respectively. Find the coordinates of A.

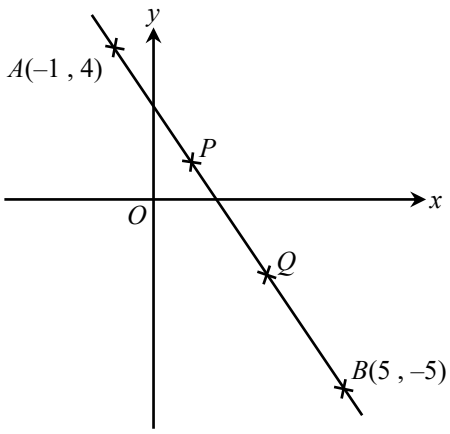


Practice Exercise

45. In the figure, $P(3, -10)$ is a point lying on the line segment joining points $A(-5, -14)$ and B such that $AP : PB = 4 : 3$. Find the coordinates of B.



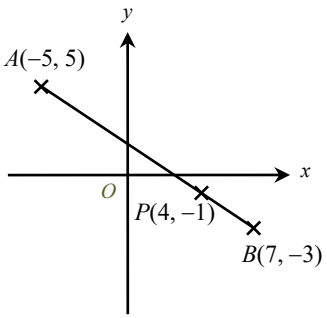
46. In the figure, the coordinates of A and B are $(-1\ ,\ 4)$ and $(5\ ,\ -5)$ respectively. P and Q are points on AB such that they divide AB into three equal parts. Find the coordinates of P and Q.



47. P is a point lying on the line segment joining points $A(a, b)$ and $B(7a, -2b)$, where a and b are constants. It is given that $AP : PB = 1 : 2$. Someone claims that if $a > 0$, P lies on the positive x-axis. Do you agree? Explain your answer.

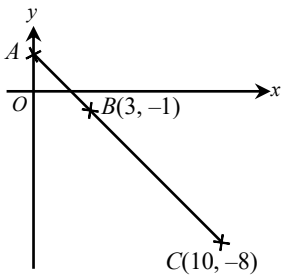
Example 12 (TB P.9.49)

In the figure, $P(4, -1)$ is a point on the line segment joining $A(-5, 5)$ and $B(7, -3)$. Find $AP : PB$.



Practice Exercise

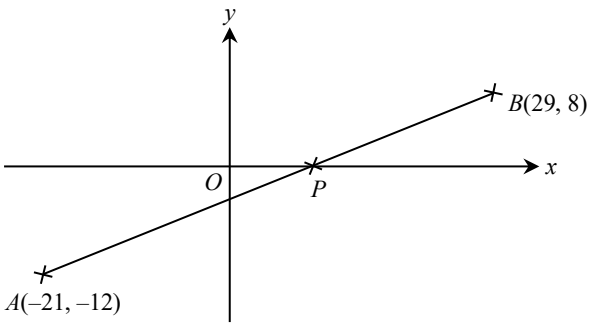
48. In the figure, $B(3, -1)$ is a point on the line segment joining the points A and $C(10, -8)$. It is given that A lies on the y-axis. Find $AB : BC$.



49. In the figure, the line segment joining points $A(-21, -12)$ and $B(29, 8)$ cuts the x-axis at point P.

(a) Find $AP : PB$.

(b) Hence, find the coordinates of P.



50. The coordinates of points A and B are (–16, 18) and (16, –22) respectively. $P(p, -12)$ is a point lying on the line segment AB, where p is a constant.

(a) Find $AP : PB$.

(b) Hence, find the value of p.

51. It is given that the line segment joining the points $A(-10, 3)$ and $B(10, -7)$ cuts the x-axis and the y-axis at points P and Q respectively.

(a) **(i)** Find $AQ : QB$.

(ii) Hence, find the coordinates of Q.

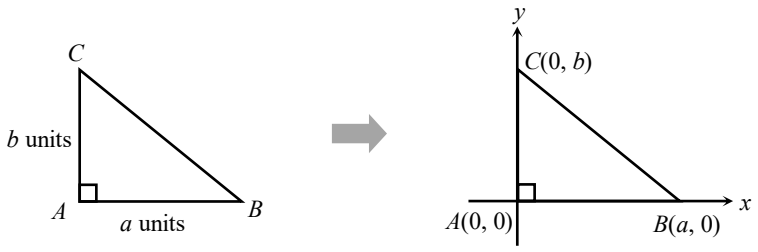
(b) Find $AP : PQ : QB$.

E Analytic Approach

Key Points: Analytic Approach

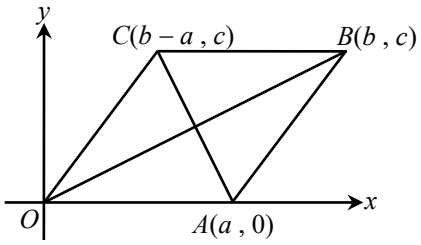
We can prove geometric results by introducing a rectangular coordinate system to the geometric figures involved.

e.g. Consider a right-angled triangle ABC with base a units and height b units.



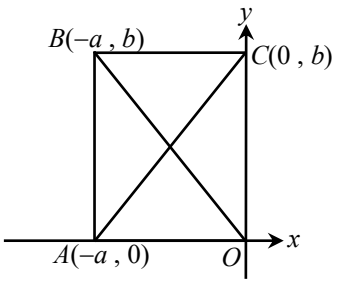
Example 13 (TB P.9.54)

In the figure, $OABC$ is a rhombus. Prove that the two diagonals of $OABC$ bisect each other.

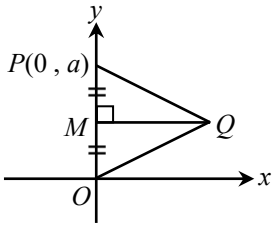


Practice Exercise

52. In the figure, $OABC$ is a rectangle. Prove that the two diagonals of $OABC$ bisect each other.



53. The figure shows $\triangle OPQ$. M is a point on OP such that MQ is the perpendicular bisector of OP. Prove that $\triangle OPQ$ is an isosceles triangle.

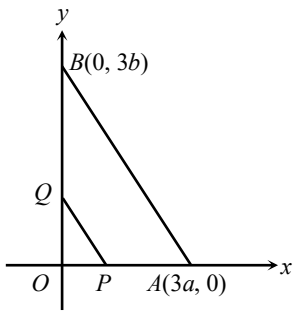


Example 14 (TB P.9.55)

In the figure, O, $A(3a, 0)$ and $B(0, 3b)$ are the vertices of $\triangle OAB$. P and Q are points on OA and OB respectively such that $OP : PA = OQ : QB = 1 : 2$.

(a) Express the coordinates of P and Q in terms of a and b.

(b) Prove that $PQ \parallel AB$.

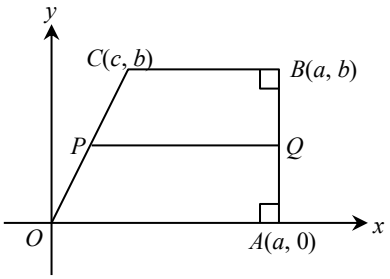


Practice Exercise

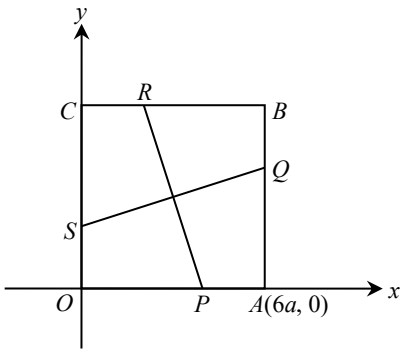
54. In the figure, $OABC$ is a trapezium with $OA \parallel CB$ and $\angle OAB = \angle ABC = 90°$. P and Q are points lying on OC and AB respectively such that $OA \parallel PQ \parallel CB$. P is the mid-point of OC.

(a) Find the coordinates of P in terms of b and c.

(b) Prove that Q is the mid-point of AB.



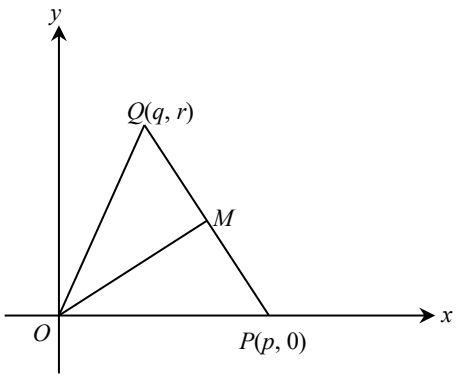
55. The figure shows a square $OABC$. P, Q, R and S are points lying on OA, AB, BC and CO respectively such that $OP : PA = AQ : QB = BR : RC = CS : SO = 2 : 1$. A rectangular coordinate system is introduced so that the coordinates of points O and A are $(0, 0)$ and $(6a, 0)$ respectively, where a is a non-zero constant. Prove that $PR \perp QS$.



56. In the figure, $OP = OQ$ and M is the mid-point of PQ.

(a) Prove that $p^2 = q^2 + r^2$.

(b) Prove that $OM \perp PQ$.



57. The figure shows $\triangle OAB$. R, Q and P are the mid-points of OA, OB and AB respectively. Prove that $PQRA$ is a parallelogram.