2024-2025 S5
1st TERM EXAM
MATH EP
M2

2024 – 2025
S5 First Term Examination

MATHEMATICS Extended Part

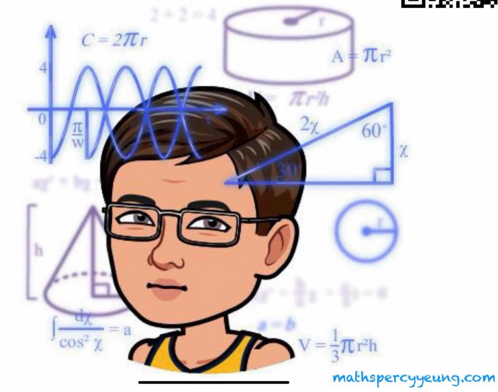
Module 2 (Algebra and Calculus)

Question–Answer Book

14th January, 2025
8:15 am – 10:15 am (2 hours)
This paper must be answered in English

INSTRUCTIONS

1. Write your name, class and class number in the spaces provided on this cover.
2. This paper consists of TWO sections, A and B.
3. Attempt ALL questions in this paper. Write your answers in the spaces provided in this Question-Answer Book. Do not write in the margins. Answers written in the margins will not be marked.
4. Unless otherwise specified, all working must be clearly shown.
5. Unless otherwise specified, numerical answers must be exact.
6. The diagrams in this paper are not necessarily drawn to scale.



Section	Marks
A Total	**/ 43**
B Total	**/ 37**
TOTAL	**/ 80**
E1	**%**

FORMULAS FOR REFERENCE

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$
$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$
$$2\sin A \cos B = \sin(A+B) + \sin(A-B)$$
$$2\cos A \cos B = \cos(A+B) + \cos(A-B)$$
$$2\sin A \sin B = \cos(A-B) - \cos(A+B)$$
$$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$$
$$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$$
$$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$$
$$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$$

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Section A (43 marks)

1. Find $\dfrac{d}{dx}\left(x^2 + \dfrac{1}{x}\right)$ from first principles. (4 marks)

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2. (a) Evaluate $\lim_{x \to \frac{\pi}{2}} \frac{x - \frac{\pi}{2}}{\cos x}$.

(b) (i) Expand $(1+mx)^n - (1+nx)^m$ in ascending powers of x as far as the term in x^3.

(ii) Hence, evaluate $\lim_{x \to 0} \frac{(1+mx)^n - (1+nx)^m}{x^2}$, where $m, n \geq 2$.

(6 marks)

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3. Find the following.

(a) $\int \sqrt{1-4x^2}\, dx$.

(b) $\int x^3 \sin(x^2+1)\, dx$.

(7 marks)

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4. (a) Using mathematical induction, prove that $\sum_{k=1}^{n} 7^{k-1} = \frac{7^n - 1}{6}$ for all positive integers n.

(b) Using (a), express $\sum_{k=m}^{2m} 7^{k+1}$ in terms of m, where m is the positive integer.

(7 marks)

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5. (a) Let n be a positive integer and $x \in \left(0, \dfrac{\pi}{n+1}\right)$.

Show that $\cot kx - \cot(k+1)x = \dfrac{\sin x}{\sin kx \sin(k+1)x}$ for all $k = 1, 2, 3, \ldots, n$.

(b) Hence, deduce that $\dfrac{1}{\sin x \sin 2x} + \dfrac{1}{\sin 2x \sin 3x} + \ldots + \dfrac{1}{\sin nx \sin(n+1)x} = \dfrac{\sin nx}{\sin^2 x \sin(n+1)x}$.

(6 marks)

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6. An inverted vessel is in the shape of a right circular cone. The base radius and the height of the vessel are 6 cm and 15 cm respectively. Let V cm^3 and h cm be the volume and the depth of the water in the vessel respectively.

(a) Express V in terms of h.

(b) Water has been leaking out of the vessel through the apex for t min. The depth of the water is given by $h = \dfrac{15}{2e^{\frac{t}{4}} + 1}$.

Find the rate of change of volume of the water in the vessel at $t = 4$.

(Give your answer correct to 2 decimal places.)

(5 marks)

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7. In Figure 1, the shaded region is bounded by the two curves $C_1 : y^2 = x - 1$, $C_2 : y^2 = -2x + 14$ and the line $y = 1$. C_1 and C_2 intersect at A in quadrant I. The line $y = 1$ intersects C_1 and C_2 at B and C respectively.

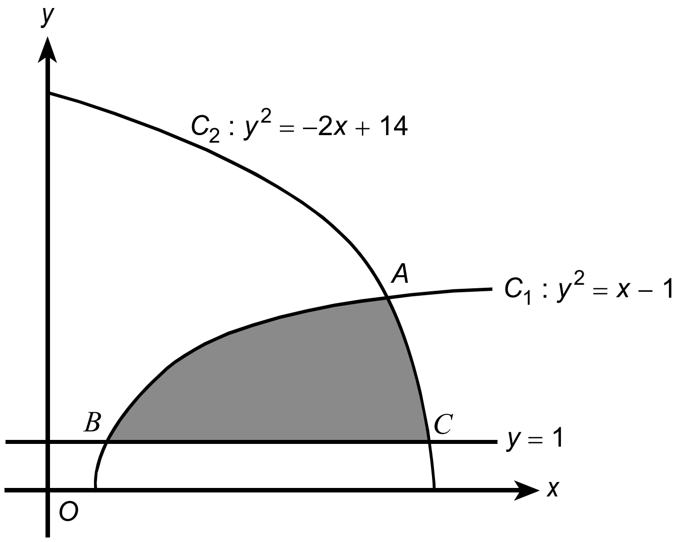

Figure 1

(a) Find the coordinates of A, B and C.

(b) If the region is revolved about the x-axis, find the volume of the solid generated.

(8 marks)

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Section B (37 marks)

8. Let $f(x)=\dfrac{x^2-x-6}{x+6}$ where $x\neq -6$.

(a) Find $f'(x)$ and $f''(x)$. (2 marks)

(b) Solve each of the following inequalities:

(i) $f'(x)>0$,

(ii) $f''(x)<0$.

(3 marks)

(c) Find the relative maximum point(s) and minimum point(s) of the graph of $y=f(x)$.

(2 marks)

(d) Find the asymptote(s) of the graph of $y=f(x)$. (2 marks)

(e) Sketch the graph of $y=f(x)$. (2 marks)

(f) Find the area of the region bounded by the graph of $y=f(x)$ and the x-axis. (3 marks)

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9. In Figure 2, a right circular cone is circumscribed to a sphere of radius 3 cm, with the base of the cone touching the sphere. Let θ be the semi-vertical angle of the cone and V cm^3 be the volume of the cone.

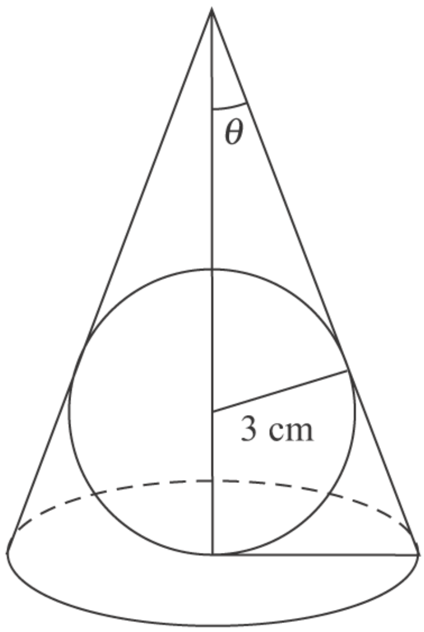

Figure 2

(a) Prove that $V = 9\pi(1+\csc\theta)^3 \tan^2\theta$. (3 marks)

(b) (i) Find $\dfrac{dV}{d\theta}$.

(ii) Find the range of values of θ such that V is decreasing.

(iii) Hence, find the minimum volume of the cone.

(9 marks)

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10. (a) Let $a > 0$ and $f(x)$ be a continuous function.

Prove that $\int_0^a f(x)dx = \int_0^a f(a-x)dx$.

Hence, prove that $\int_0^a f(x)dx = \frac{1}{2}\int_0^a \left[f(x) + f(a-x) \right] dx$. (3 marks)

(b) Show that $\int_0^1 \frac{dx}{x^2 - x + 1} = \frac{2\sqrt{3}\,\pi}{9}$. (4 marks)

(c) Using **(a)** and **(b)**, or otherwise, evaluate $\int_0^1 \frac{dx}{\left(x^2 - x + 1\right)\left(e^{2x-1} + 1\right)}$. (4 marks)

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