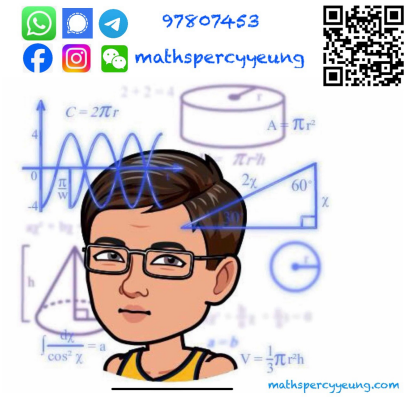
2024-2025 S6
MOCK EXAM
MATH EP
M2



2024 – 2025
S6 Mock Examination

MATHEMATICS Extended Part

Module 2 (Algebra and Calculus)

Question–Answer Book

11th February, 2025
8:15 am – 10:45 am (2 hours 30 minutes)
This paper must be answered in English

INSTRUCTIONS

1. Write your name, class and class number in the spaces provided on this cover.

2. This paper consists of TWO sections, A and B.

3. Attempt ALL questions in this paper. Write your answers in the spaces provided in this Question – Answer Book. Do not write in the margins. Answers written in the margins will not be marked.

4. Unless otherwise specified, all working must be clearly shown.

5. Unless otherwise specified, numerical answers must be exact.

6. The diagrams in this paper are not necessarily drawn to scale.

Section	Marks
A Total	**/50**
B Total	**/50**
TOTAL	**/100**

FORMULAS FOR REFERENCE

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$	$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$
$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$	$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$
$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$	$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$
$2\sin A \cos B = \sin(A+B) + \sin(A-B)$	$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$
$2\cos A \cos B = \cos(A+B) + \cos(A-B)$	
$2\sin A \sin B = \cos(A-B) - \cos(A+B)$	

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Section A (50 marks)

1. Let $f(x) = \csc x$. Find $f'(x)$ from the first principles. Hence, find $f'\left(\frac{\pi}{4}\right)$. (5 marks)

Answers written in the margins will not be marked

2. Let a be a constant and n be a positive integer.

(a) Expand $(1+ax)^9-(1+3x)^n$ in ascending powers of x up to the x^2 term.

(b) If the coefficients of x and x^2 in (a) are -57 and 387 respectively, find a and n.

(6 marks)

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3. (a) Prove that $\cos 3x = 4\cos^3 x - 3\cos x$.

(b) Let $\frac{\pi}{2} < x < \pi$.

(i) Prove that $\frac{\cos 3\left(\frac{5\pi}{6} - x\right)}{\cos\left(\frac{5\pi}{6} - x\right)} = \frac{2\sin 3x}{\sin x - \sqrt{3}\cos x}$.

(ii) Solve the equation $\frac{2\sin 3x}{\sin x - \sqrt{3}\cos x} = -1$.

(7 marks)

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4. (a) Using mathematical induction, prove that

$$\sum_{k=1}^{2n} 3r^2 = n(2n+1)(4n+1) \qquad \text{for all positive integers } n .$$

(b) Using (a), evaluate $\sum_{r=11}^{42} 3r^2$.

(6 marks)

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5. M is a point lying on XY such that $XM : MY = 1 : 4$. Let $\overrightarrow{OX} = \mathbf{x}$ and $\overrightarrow{OY} = \mathbf{y}$. It is given that $|\mathbf{x}| = 3$, $|\mathbf{y}| = 5$ and $\cos \angle XOY = -\frac{1}{6}$.

(a) Find $\mathbf{x} \cdot \mathbf{y}$.

(b) Find the unit vector in the direction of $\left|\overrightarrow{OM}\right|$ in terms of $\mathbf{x}$ and $\mathbf{y}$.

(6 marks)

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6. Define $h(x) = \dfrac{\ln x}{\sqrt{x}}$ for all $x \in (0, 99)$. Denote the graph of $y = h(x)$ by G.

(a) Prove that G has only one maximum point.

(b) Let R be the region bounded by G, the x-axis and the vertical line passing through the maximum point of G. Find the volume of the solid of revolution generated by revolving R about x-axis.

(6 marks)

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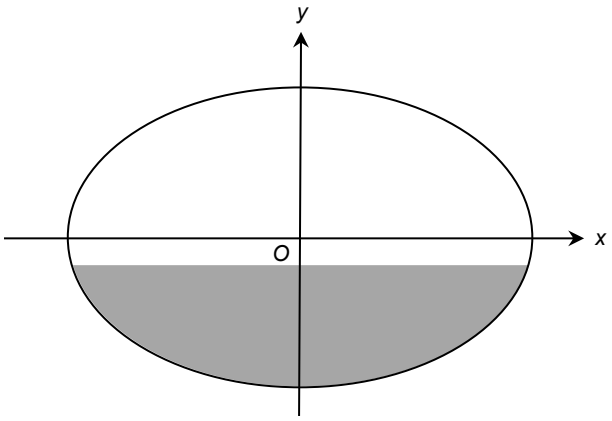
7. (a) In the figure, the shaded region is bounded by the curve $\frac{x^2}{25}+\frac{y^2}{16}=1$, and the line $y=-h$, where $0 < h < 4$. A solid is generated by revolving the region about the y-axis. Prove that the volume of the solid of revolution is $25\pi\left(\frac{8}{3}-h+\frac{h^3}{48}\right)$.



(3 marks)

(b) An empty bowl is in the shape of the solid describe in (a) with height 4 cm. Water is poured into the cup at a rate of π cm^3/s.

(i) Find the time elapsed when the depth is 3 cm.

(ii) Find the rate of increase of the depth of water when the depth is 3 cm.

(4 marks)

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8. Consider the system of linear equations in real variables x, y, z

$$(E): \begin{cases} x + (\alpha-2)y - z = \beta \\ 2x + y + \alpha z = -6\beta^2 \\ 2x + (\alpha+3)y - 2\alpha z = 8\beta \end{cases}, \text{ where } \alpha, \beta \in \mathbf{R}.$$

(a) Assume that (E) has a unique solution.

(i) Show that $\alpha \neq \frac{1}{5}$ and $\alpha \neq 4$.

(ii) Express y in terms of α and β.

(b) Assume that $\alpha = 4$ and (E) is inconsistent. Find the range of values of β.

(7 marks)

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Section B (50 marks)

9. Define $f(x)=\dfrac{x^2+ax+10}{x-6}$, for all $x\neq 6$. Denote the graph of $y=f(x)$ as G. The line $y=x-1$ is an oblique asymptote of G.

(a) Find the value of a and the remaining asymptote(s). (3 marks)

(b) Find $f'(x)$. (2 marks)

(c) Find the maximum point(s) and the minimum point(s) of G. (4 marks)

(d) Let R be the region bounded by G and the x-axis. Find the area of R. (3 marks)

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10. (a) Rewrite $3x^2+2x+5$ in the form of $a[(x+b)^2+c]$, where a, b and c are constants.

(1 mark)

(b) Using $\tan^{-1}\left(\frac{4}{\sqrt{14}}\right)-\tan^{-1}\left(\frac{1}{\sqrt{14}}\right)=\tan^{-1}\left(\frac{7}{3\sqrt{14}}\right)$, evaluate $\int_0^1 \frac{dx}{3x^2+2x+5}$. (3 marks)

(c) (i) Let $t=\tan\frac{x}{2}$, where $0\le x\le\frac{\pi}{2}$. Prove that $\sin x=\frac{2t}{1+t^2}$ and $\cos x=\frac{1-t^2}{1+t^2}$.

(ii) Using the substitution $t=\tan\frac{x}{2}$, evaluate $\int_0^{\frac{\pi}{2}} \frac{1}{\sin x+\cos x+4}dx$.

(5 marks)

(d) Prove that $\int_0^{\frac{\pi}{2}} \frac{\sin x+2}{\sin x+\cos x+4}dx=\int_0^{\frac{\pi}{2}} \frac{\cos x+2}{\sin x+\cos x+4}dx$. (2 marks)

(e) Evaluate $\int_0^{\frac{\pi}{2}} \frac{4\sin x+9}{\sin x+\cos x+4}dx$. (3 marks)

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11. Let $M = \begin{pmatrix} -4 & -6 \\ 1 & 3 \end{pmatrix}$. Denote the 2×2 identity matrix by I.

(a) Find a pair of real numbers a and b such that $M^2 = aM + bI$. (3 marks)

(b) Prove that $5M^n = [2^n - (-3)^n]M + [3(2^n) + 2(-3)^n]I$ for all positive integers n. (4 marks)

(c) If $(M^{-1})^n = cM + dI$, where n is any positive integer and c, $d \in \mathbf{R}$, find c and d in terms of n. (5 marks)

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12. The position vectors of the points A, B, C and D are $t\mathbf{i}+4\mathbf{j}+s\mathbf{k}$, $8\mathbf{i}+2t\mathbf{j}+2s\mathbf{k}$, $-2t\mathbf{i}+(2-s)\mathbf{j}+14\mathbf{k}$ and $(t-4)\mathbf{i}+s\mathbf{j}-(s+t)\mathbf{k}$ respectively, where $s,t\in\mathbf{R}$. Suppose that $\overrightarrow{AB}$ is parallel to $2\mathbf{i}-2\mathbf{j}+\mathbf{k}$. Denote the plane which contains A, B and C by Π.

(a) Find

(i) s and t,

(ii) the area of ΔABC.

(5 marks)

(b) Let E be the projection of D on Π.

(i) Find $\overrightarrow{DE}$.

(ii) Find the volume of the tetrahedron $ABCD$.

(iii) Someone claims that E is the orthocentre of ΔABC. Do you agree? Explain your answer.

(7 marks)

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