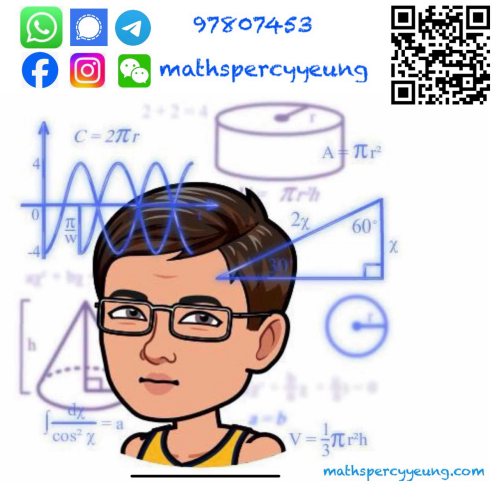
2023-2024 S6
MOCK EXAM
MATH EP
M2



2023 – 2024
S6 Mock Examination

MATHEMATICS Extended Part

Module 2 (Algebra and Calculus)

Question–Answer Book

30th January, 2024
8:15 am – 10:45 am (2 hours 30 minutes)
This paper must be answered in English

INSTRUCTIONS

1. Write your name, class and class number in the spaces provided on this cover.

2. This paper consists of TWO sections, A and B.

3. Attempt ALL questions in this paper. Write your answers in the spaces provided in this Question – Answer Book. Do not write in the margins. Answers written in the margins will not be marked.

4. Unless otherwise specified, all working must be clearly shown.

5. Unless otherwise specified, numerical answers must be exact.

6. The diagrams in this paper are not necessarily drawn to scale.

Section	Marks
A Total	**/50**
B Total	**/50**
TOTAL	**/100**

FORMULAS FOR REFERENCE

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$	$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$
$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$	$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$
$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$	$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$
$2\sin A \cos B = \sin(A+B) + \sin(A-B)$	$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$
$2\cos A \cos B = \cos(A+B) + \cos(A-B)$	
$2\sin A \sin B = \cos(A-B) - \cos(A+B)$	

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Section A (50 marks)

1. Let $f(x) = x \cos 2x$.
   (a) Prove that $f(\pi + h) - f(\pi) = -2\pi \sin^2 h + h \cos 2h$.
   (b) Using (a), find $f'(\pi)$ from first principles.

(5 marks)

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2. Let a be a non-zero constant. If the sum of the coefficient of x^3 and the constant term in the expansion of $(x+1)^4\left(-4x+\frac{a}{x^2}\right)^5$ is $-640a$, find a.

(5 marks)

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3. P is a point lying on AB such that $AP : PB = 1 : 3$. Let $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$, where O is the origin.

(a) Express $\overrightarrow{OP}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

(b) It is given that $|\mathbf{a}| = \sqrt{2}$, $|\mathbf{b}| = 2$ and $|\overrightarrow{OP}| = \sqrt{\frac{17}{8}}$.

(i) Find $\mathbf{a} \cdot \mathbf{b}$.

(ii) $\angle AOB$.

(6 marks)

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4. (a) Prove that $\cos^2 x - \sin^2 y = \cos(x+y)\cos(x-y)$.

(b) Solve the equation $\cos^2 2\theta - \sin^2 3\theta + \cos\theta \sin 5\theta = 0$, where $0 \le \theta \le \frac{\pi}{2}$.

(6 marks)

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5. In Figure 1, OA is an inclined wall on the horizontal ground OB, where $\angle AOB = \frac{\pi}{3}$. PQ is a rod of length $\sqrt{13}$ m, where P and Q lie on OA and OB respectively. Let $OP = x$ m and $OQ = y$ m at time t s.

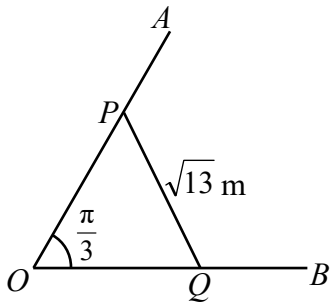

Figure 1

(a) Show that $\frac{dy}{dt} = \left(\frac{2x-y}{x-2y}\right)\frac{dx}{dt}$.

(b) P moves towards O along OA at a constant speed of $\frac{1}{2}$ m/s. When P is at a distance of 3 m from O, find the rate of change of the distance of Q from O.

(7 marks)

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6. (a) Find a pair of constants a and b such that

$$13\cos x - 6\sin x \equiv a(\sin x + 2\cos x) + b(\cos x - 2\sin x).$$

(b) Evaluate $\int_0^{\frac{\pi}{4}} \frac{13-6\tan x}{\tan x+2}\,dx$.

(6 marks)

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7. Let $A = \begin{pmatrix} 4 & 0 \\ -3 & 1 \end{pmatrix}$ and n be a positive integer.

(a) Define $P = \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}$. Evaluate $P^{-1}AP$.

(b) Evaluate A^n .

(c) Let α be a real number greater than 1 . Denote the 2×2 identity matrix by I . If $A^n - \alpha I$ is a singular matrix, express α in terms of n .

(7 marks)

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8. (a) Using mathematical induction, prove that

$$\sum_{k=1}^{n} \frac{1}{(2k-1)(2k+1)(2k+3)} = \frac{1}{12} - \frac{1}{4(2n+1)(2n+3)}$$

for all positive integers n.

(b) Using (a), evaluate $\sum_{k=3}^{12} \frac{1}{(2k-1)(2k+1)(2k+3)}$.

(8 marks)

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Section B (50 marks)

9. (a) Find $\int \frac{1}{2x^2+1}\,dx$.

(2 marks)

(b) Define $f(x)=\frac{2(2x+1)(x-1)^2}{2x^2+1}$ for all real numbers x . Denote the graph of $y=f(x)$ by G .

(i) Find the asymptote of G .

(ii) Find the maximum point(s) and minimum point(s) of G .

(iii) Find the area of the region bounded by G, the y-axis and the straight line $y = 2x - 3$ for $x \geq 0$.

(10 marks)

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10. Let O be the origin. The position vectors of the points A and B are $7\mathbf{i} + 3\mathbf{j} - 5\mathbf{k}$ and $-3\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$ respectively. C is a point on AB such that $AC : CB = 2 : 3$. Let Π be the plane containing O, B and C.

(a) Find $\overrightarrow{OC}$.

(2 marks)

(b) Find a non-zero vector which is perpendicular to Π.
Hence, find the area of ΔOBC.

(4 marks)

(c) It is given that the position vector of the point D is $2\mathbf{i} - 4\mathbf{j}$. Denote the projection of D on Π by E.

(i) Describe the geometric relationship between B, C and E. Explain your answer.

(ii) Find the angle between OD and the plane BCD correct to 3 significant figures.

(6 marks)

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11. (a) Consider the system of linear equations in real variables x, y, z

$$(E): \begin{cases} x + 2y - z = -2 \\ 4x + 6y + az = b \\ 2x + (a-2)y + (a+2)z = b+8 \end{cases}, \text{ where } a, b \in \mathbf{R}.$$

(i) Assume that (E) has a unique solution. Find the range of values of a.

(ii) Assume that $a = -4$. If (E) is consistent, find b.

(iii) Assume that $a \neq -4$ and (E) is inconsistent. Find the range of values of b.

(7 marks)

(b) Consider the system of linear equations in real variables x, y, z

$$(F): \begin{cases} x + 2y - z = -2 \\ 4x + 6y + z = s \\ 2x - y + 3z = s+8 \end{cases}, \text{ where } s \in \mathbf{R}.$$

Does there exist a pair of real constants m and n (independent of s) such that for every $s \in \mathbf{R}$, (F) has a real solution (x, y, z) satisfying $mx + y + nz = -\frac{1}{3}$? Explain your answer.

(6 marks)

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12. (a) Let a be a non-zero constant. Prove that $\int_0^1 x\,e^{ax}\,dx = \dfrac{a\,e^{a} - e^{a} + 1}{a^{2}}$.

(3 marks)

(b) Using (a) and integration by substitution, evaluate $\int_0^{e-1} \dfrac{\ln(x+1)}{(1+x)^{r}}\,dx$, where r is a constant and $r \neq 1$.

(4 marks)

(c) Evaluate $\int_0^{\frac{\pi}{4}} \dfrac{\ln[1+(e-1)\tan x]}{(1+e\tan x-\tan x)^{3}}\,dx + \int_0^{\frac{\pi}{4}} \dfrac{\tan^{2} x\ \ln[1+(e-1)\tan x]}{(1+e\tan x-\tan x)^{3}}\,dx$.

(3 marks)

(d) Evaluate $\int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \dfrac{\csc^{2} 2x\ \ln[1+(e-1)\cot 2x]}{(1+e\cot 2x-\cot 2x)^{3}}\,dx$.

(3 marks)

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