(*) Out syl after HKDSE 2022: normal of the curve

Chapter 8

Applications of Differentiation

Exercise 8A

Level 1

1. Find the slope of the tangent to the curve $y = 3x^2 + 4x + 5$ at $x = -3$.

(*) 2. Find the slope of the normal to the curve $y = (x^2 - 2)^2$ at $x = 3$.

3. Find the slope of the tangent to the curve $y = \frac{x}{2} + 2 + \frac{2}{x}$ at $(-2, 0)$.

(*) 4. Find the slope of the normal to the curve $y = x\sqrt{x^2 - 4x + 9}$ at $(4, 12)$.

(*) 5. Find the slope of the normal to the curve $2x^3 - y^2 = 4y - x$ at $(0, -4)$.

6. Find the slope of the tangent to the curve $3x^3y + 2x^2y^2 = 2$ at $(-1, 2)$.

7. Find the slope of the tangent to the curve $y = \tan 3x$ at $(\frac{\pi}{4}, -1)$.

(*) 8. Find the equations of the tangent and the normal to the curve $y = -x^2 + 6x + 18$ at $(-2, 2)$.

(*) 9. Find the equations of the tangent and the normal to the curve $y = 2x^3 + 2x^2 - 4x - 8$ at $(-1, -4)$.

(*) 10. Find the equations of the tangent and the normal to the curve $y = \frac{3x}{x - 3}$ at $(-6, 2)$.

(*) 11. Find the equations of the tangent and the normal to the curve $y = \frac{4}{(1 - 2x)^2}$ at $(1, 4)$.

(*) 12. Find the equations of the tangent and the normal to the curve $y = \sqrt{2x^2 - 6x + 8}$ at $(1, 2)$.

(*) 13. Find the equations of the tangent and the normal to the curve $y = \sqrt{\frac{x + 4}{x - 4}}$ at $(5, 3)$.

(*) 14. Find the equations of the tangent and the normal to the curve $y = \frac{2x + 6}{2x - 3}$ at $x = 2$.

(*) 15. Find the equations of the tangent and the normal to the curve $y = \frac{1}{\sqrt{4x^2 + 2x + 9}}$ at $x = -\frac{1}{2}$.

(*) 16. Find the equations of the tangent and the normal to the curve $3y^3 - 4x^2y + 2xy = 0$ at $(2, -2)$.

17. If $f(x)=\dfrac{1+2e^x}{1-2e^x}$, find

(a) $f'(0)$.

(b) the equation of the tangent to the curve $y=f(x)$ at $(0, -3)$.

(*) **18.** If $f(x)=\dfrac{e^x+e^{-x}}{x+1}$, find

(a) $f'(0)$.

(b) the equations of the tangent and the normal to the curve $y=f(x)$ at $x=0$.

(*) **19.** If $f(x)=xe^x$, find

(a) $f'(e)$.

(b) the equations of the tangent and the normal to the curve $y=f(x)$ at $x=e$.

(*) **20.** If $f(x)=e^{x-1}\ln(2x+e-2)$, find

(a) $f'(1)$.

(b) the equations of the tangent and the normal to the curve $y=f(x)$ at $x=1$.

(*) **21.** Find the equations of the tangent and the normal to the curve $y=\sin x\cos x$ at $(\frac{2\pi}{3}, -\frac{\sqrt{3}}{4})$.

(*) **22.** Find the equations of the tangent and the normal to the curve $x=\sqrt{y^2+4y+11}$ at $(4, 1)$.

(*) **23.** Find the equations of the tangent and the normal to the curve $x=\dfrac{\sqrt{y}+1}{y-1}$ at $(1, 4)$.

24. Find the equations of the tangent to the curve $y=3x^2+8x+4$ where the slope of the tangent is 2.

(*) **25.** Find the equations of the normals to the curve $y=9x^3+18x^2-34x-20$ where the normals are parallel to the straight line $x+2y+1=0$.

26. Find the equation of the tangent to the curve $y=-x^2-6x-8$ where the tangent is perpendicular to the straight line $x-2y+6=0$.

27. Find the equations of the tangents to the curve $y=x^2+2x-2$ where the tangents pass through $(-1, -7)$.

28. Find the equations of the tangents to the curve $y=-x^2+6x+2$ where the tangents pass through $(-1, 4)$.

Level 3

29. It is given that (0, 1) and (1, 4) are two points on the curve $y = -x^3 + ax^2 + bx + c$, where a, b and c are constants. If the slope of the tangent to the curve at (1, 4) is -1, find the values of a, b and c.

30. **(a)** Express the equation of the tangent to the curve $y = 2x^2 + 4px + 5$ in terms of p where the tangent is parallel to the straight line $16x - y = 0$ and p is a constant.

(b) If the y-intercept of the tangent obtained in **(a)** is -13, find the values of p.

31. Find the equation of the tangent to the curve $y = \ln 2x^2 + x + 2$ where the tangent is parallel to the straight line $4x - 2y - 5 = 0$.

32. Find the equations of the tangents to the curve $y = \dfrac{e^{6x^2}}{x^2}$ where the tangents pass through the origin.

33. Find the point at which the tangent to the curve $y = x^3 - 3x^2$ at (2, −4) meets the curve again.

34. It is given that the curve $y = x^3 - 3x + p$ passes through $A(\frac{1}{2}, q)$ and $B(-1, 8)$, where p and q are constants.

(a) Find the values of p and q.

(b) Find the slope of AB.

(c) It is given that C is a point on the curve (A, B and C are three distinct points) such that the tangent to the curve at C is parallel to AB. Find the equation of the tangent to the curve at C.

EXERCISE 8B

Level 1

35. Find the range(s) of values of x such that $f(x) = -x^2 + 4x - 7$ is **(a)** increasing, **(b)** decreasing.

36. Find the range(s) of values of x such that $f(x) = x^3 + 3x^2 - 24x + 15$ is **(a)** increasing, **(b)** decreasing.

37. Find the range(s) of values of x such that $f(x) = -4x^3 - 12x^2 - 9x + 1$ is **(a)** increasing, **(b)** decreasing.

38. Use the information provided to find the maximum and minimum points of the curve $y = f(x)$.

x	$x < -3$	$x = -3$	$-3 < x < 1$	$x = 1$	$x > 1$
$f(x)$	↘	-4	↗	3	↘
$f'(x)$	−	0	+	0	−

39. Use the information provided to find the maximum and minimum points of the curve $y = f(x)$.

x	$x < 0$	$x = 0$	$0 < x < 2$	$2 < x < 4$	$x = 4$	$x > 4$
$f(x)$	/	8	/	/	12	/
$f'(x)$	+	0	–	–	0	+

40. Find the local extrema of $f(x) = -x^3 + 2x^2 + 4x + 2$.

41. Find the local extrema of $f(x) = x^3 - 3x^2 - 9x + 12$.

42. Find the local extrema of $f(x) = x^2(x^2 - 2) - 1$.

43. Find the local extrema of $f(x) = -x^4 - 4x^3 + 8x^2 + 48x + 20$.

44. Find the local extrema of $f(x) = \dfrac{x^2 - 2x + 4}{x^2 + 4}$.

45. Find the local extrema of $f(x) = 2x^3 - 6x^2 + 6x + 6$.

Level 2

46. Find the turning point(s) of the curve $y = (x + 2)^8 + 2$.

47. Find the turning point(s) of the curve $y = (x + 1)^4(x - 2)^2$.

48. Find the turning point(s) of the curve $y = \dfrac{x}{(x + 5)^2 + 11}$.

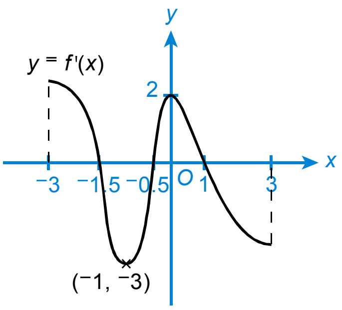
49. Find the turning point(s) of the curve $y = \dfrac{10(x + 2)}{x^2 + 5}$.

50. Find the turning point(s) of the curve $y = \dfrac{x^2 + 9}{x^2 + 3x + 9}$.

51. Find the turning point(s) of the curve $y = \sqrt{x^2 + 8x + 20}$.

52. Find the turning point(s) of the curve $y = x^5(x^2 + 7) + 3$.

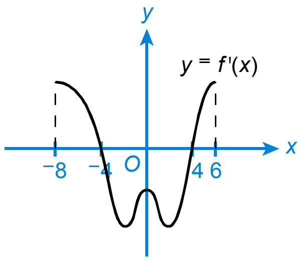
53. Let $f(x)$ be a continuous function for $-3 \le x \le 3$. The figure shows the graph of $y = f'(x)$.



(a) Write down the range(s) of values of x such that $f(x)$ is increasing.

(b) Find the x-coordinates of the maximum point(s) and the minimum point(s) of the curve $y = f(x)$.

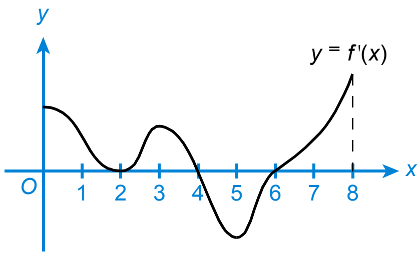
54. Let $f(x)$ be a continuous function for $-8 \le x \le 6$. The figure shows the graph of $y = f'(x)$.



(a) Write down the range(s) of values of x such that $f(x)$ is increasing.

(b) Find the x-coordinates of the maximum point(s) and the minimum point(s) of the curve $y = f(x)$.

55. Let $f(x)$ be a continuous function for $0 \le x \le 8$. The figure shows the graph of $y = f'(x)$.



(a) Write down the range(s) of values of x such that $f(x)$ is decreasing.

(b) Find the x-coordinates of the maximum point(s) and the minimum point(s) of the curve $y = f(x)$.

56. **(a)** Find the turning point(s) of the curve $y = \dfrac{x^2 + 5}{x - 2}$.

(b) Must the local maximum of a function be greater than its local minimum? Explain briefly.

57. **(a)** Find the turning point(s) of the curve $y = \dfrac{x + 2}{x^2 - 3}$.

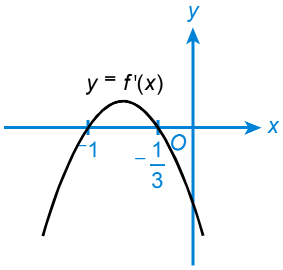
(b) Must the local maximum of a function be greater than its local minimum? Explain briefly.

58. Let $f(x) = x^3 + kx^2 - 3kx + 15$, where k is a negative constant. It is known that there is only a value of x, i.e. $x = x_0$, such that $f'(x_0) = 0$.

(a) Find the values of k and x_0.

(b) Is $f(x_0)$ a local extremum? Explain briefly.

59. Let $f(x) = -x(x-p)^2 + qx^2$, where p and q are constants. The figure shows the graph of $y = f'(x)$.



(a) Express $f'(x)$ in terms of p, q and x.

(b) From the figure, find the values of p and q.

60. The curve C: $y = -x^3 + px^2 + qx + r$, where p, q and r are constants, touches the x-axis at $x = 2$ and cuts the x-axis at $x = -1$.

(a) Find the values of p, q and r.

(b) Find the turning point(s) of C.

(c) Find the equation of the tangent to C at $x = -2$.

EXERCISE 8C

Level 1

61. Let $f(x) = x^3 - 6x^2 - 7x + 9$. Find the point(s) of inflexion of the curve $y = f(x)$.

62. Let $f(x) = -2x^3 - 24x^2 - 11x + 220$. Find the point(s) of inflexion of the curve $y = f(x)$.

63. Find the point(s) of inflexion of the curve $y = 5(x-2)^3$.

64. Find the point(s) of inflexion of the curve $y = 3(2x-1)^3$.

65. Find the point(s) of inflexion of the curve $y = x^4 - 24x^2 - 7x + 63$.

66. Find the point(s) of inflexion of the curve $y = (x-1)(2x+3)^2$.

67. For the curve $y = -2x^3 + 24x + 7$, find the stationary points and determine whether they are maximum points, minimum points or points of inflexion.

68. For the curve $y = (x-1)^2(x+1)$, find the stationary points and determine whether they are maximum points, minimum points or points of inflexion.

69. For the curve $y = 7 - 2x^3$, find the stationary points and determine whether they are maximum points, minimum points or points of inflexion.

70. For the curve $y = 2(x+2)^3 - 1$, find the stationary points and determine whether they are maximum points, minimum points or points of inflexion.

Level 2

71. Use the information provided to find the turning point(s) and point(s) of inflexion of the graph of the continuous function $y = f(x)$.

	$f(x)$	$f'(x)$	$f''(x)$
$x < -2$		–	+
$x = -2$	1	0	+
$-2 < x < 1$		+	+
$x = 1$	2	+	0
$1 < x < 9$		+	–
$x = 9$	5	0	–
$x > 9$		–	–

72. Use the information provided to find the turning point(s) and point(s) of inflexion of the graph of the continuous function $y = f(x)$.

	$f(x)$	$f'(x)$	$f''(x)$
$x < -3$		–	+
$x = -3$	–2	0	+
$-3 < x < -2$		+	+
$x = -2$	0	+	0
$-2 < x < -1$		+	–
$x = -1$	1	0	–
$-1 < x < 0$		–	–
$x = 0$	0	–	0
$0 < x < 1$		–	+
$x = 1$	–2	0	+
$x > 1$		+	+

73. For the graph of the function $y = x^3 + 9x^2 + 24x + 40$, find the turning point(s) and point(s) of inflexion.

74. For the graph of the function $y = x^4 - 12x^2 + 20$, find the turning point(s) and point(s) of inflexion.

75. For the graph of the function $y = (2x+1)^3(2x-1)$, find the turning point(s) and point(s) of inflexion.

76. For the graph of the function $y = \dfrac{2}{x^2+3}$, find the turning point(s) and point(s) of inflexion.

77. For the graph of the function $y = \dfrac{1}{(x^2+7)^3}$, find the turning point(s) and point(s) of inflexion.

78. For the graph of the function $y = \dfrac{x^2}{x^2+9}$, find the turning point(s) and point(s) of inflexion.

79. For the graph of the function $y = \dfrac{x}{(x^2+3)^2}$, find the turning point(s) and point(s) of inflexion.

EXERCISE 8D

Level 1

80. Find the vertical asymptote(s) of the graph of the function $f(x) = \dfrac{5}{x+3}$.

81. Find the vertical asymptote(s) of the graph of the function $f(x) = \dfrac{7}{x^2+13x+42}$.

82. Find the vertical asymptote(s) of the graph of the function $f(x) = \dfrac{3-x^2}{x^2-9}$.

83. Find the vertical asymptote(s) of the graph of the function $f(x) = \dfrac{x^2+x-2}{x^2+2x-3}$.

84. Find the horizontal asymptote of the graph of the function $f(x) = \dfrac{x^2-4}{x^2}$.

85. Find the horizontal asymptote of the graph of the function $f(x) = \dfrac{x^5}{7-x^5}$.

86. Find the horizontal asymptote of the graph of the function $f(x) = \dfrac{x^4-x^3+x^2-x+1}{2x^5-3x^4+4x^3+5x^2}$.

87. Find the horizontal asymptote of the graph of the function $f(x) = \dfrac{(x+1)(2x-1)(x+3)}{(3x-2)(x+2)(2x-5)}$.

88. Find the horizontal asymptote of the graph of the function $f(x) = \dfrac{(3-x)(2+x)}{(4-3x)(x-5)(2x+3)}$.

89. Find the oblique asymptote of the graph of the function $f(x) = \dfrac{x-1}{x^2+4} - 3 + 4x$.

90. Find the oblique asymptote of the graph of the function $f(x) = \dfrac{x^2-3x+4}{x-1}$.

91. Find the oblique asymptote of the graph of the function $f(x) = -\dfrac{x^3 - 2x^2 - x + 1}{(x-2)(x+1)}$.

92. Find the oblique asymptote of the graph of the function $f(x) = \dfrac{2x^3 + x^2 - 12x + 11}{(x-1)(x+3)}$.

Level 2

93. Find all asymptotes of the graph of the function $f(x) = \dfrac{x^2 + 1}{x - 1}$.

94. Find all asymptotes of the graph of the function $f(x) = \dfrac{x^3 + 1}{x^2 - 9}$.

95. A curve $y = -x^4 + x^2 + 1$ is given.

(a) Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$.

(b) Find the turning point(s) and point(s) of inflexion of the curve.

(c) Sketch the curve.

96. A curve $y = \dfrac{x^2}{25 - x^2}$ is given.

(a) Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$.

(b) Find the turning point(s) and point(s) of inflexion of the curve.

(c) Find the asymptote(s) of the curve.

(d) Sketch the curve.

97. A curve $y = \dfrac{x^2 + x + 1}{x^2 + 1}$ is given.

(a) Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$.

(b) Find the turning point(s) and point(s) of inflexion of the curve.

(c) Find the asymptote(s) of the curve.

(d) Sketch the curve.

98. A curve $y = x + \dfrac{1}{x} - 1$ is given.

(a) Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$.

(b) Find the turning point(s) and point(s) of inflexion of the curve.

(c) Find the asymptote(s) of the curve.

(d) Sketch the curve.

99. A curve $y = \dfrac{2x^3}{(x-2)^2}$ is given.

(a) Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$.

(b) Find the turning point(s) and point(s) of inflexion of the curve.

(c) Find the asymptote(s) of the curve.

(d) Sketch the curve.

100. A function $f(x) = \dfrac{x}{x^2+4}$ is given.

(a) Find $f'(x)$ and $f''(x)$.

(b) Find the turning point(s) and point(s) of inflexion of the graph of the function.

(c) Find the asymptote(s) of the curve $y = f(x)$.

(d) Consider $f(-x)$ and $f(x)$. Is the graph of $y = f(x)$ symmetrical about the y-axis?

(e) Sketch the graph of $y = f(x)$.

101. A curve $y = \dfrac{x^2}{x^2+9}$ is given.

(a) Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$.

(b) Find the turning point(s) and point(s) of inflexion of the curve.

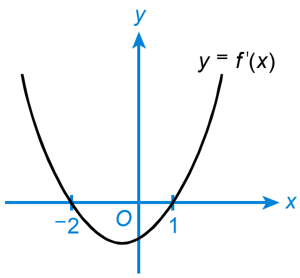
(c) Find the asymptote(s) of the curve.

(d) Let $y = f(x)$ and consider $f(-x)$ and $f(x)$. Is the curve $y = f(x)$ symmetrical about the y-axis?

(e) Sketch the curve.

Level 3

102. A function $f(x) = ax^3 + bx^2 - 6x$ is given, where a and b are constants. The figure shows a sketch of $y = f'(x)$.



(a) Find the values of a and b.

(b) Find the y- and x-intercepts of the graph of the function $y = f(x)$.

(c) Find the turning point(s) and point(s) of inflexion of the graph of the function $y = f(x)$.

(d) Sketch the graph $y = f(x)$.

EXERCISE 8E

Level 1

103. Find the global extrema of $f(x) = x^2 + 6x - 2$ on $-5 \le x \le 0$.

104. Find the global extrema of $f(x) = -(x+2)^2 + 5$ on $-4 \le x \le 2$.

105. Find the global extrema of $f(x) = 5 - 4x + 2x^2$ on $-2 \le x \le 3$.

106. Find the global extrema of $f(x) = -x^2 - 4x - 4$ on $-5 \le x \le -1$.

107. Find the global extrema of $f(x) = (x-1)^2 + 4$ on $-\infty < x < \infty$.

108. Find the global extrema of $f(x) = -(2x+3)^2 + 4x$ on $-\infty < x < \infty$.

109. Find the global extrema of $f(x) = 2x^3 - 54x$ on $0 < x < 6$.

110. Find the global extrema of $f(x) = -x^3 - \frac{9}{2}x^2 + 12x + 6$ on $-5 < x < 0$.

111. Find the global extrema of $f(x) = x^3 + 3x^2 - 9x + 9$ on $-6 \le x \le 2$.

112. Find the global extrema of $f(x) = 4x^3 - 9x^2 - 12x + 17$ on $-2 \le x \le 3$.

113. Find the global extrema of $f(x) = 2x^4 - x^2$ on $0 < x < \infty$.

114. Find the global extrema of $f(x) = x^3 - 3x^2 + 3x - 3$ on $-\infty < x < \infty$.

115. Find the global extrema of $f(x) = 4x^4 + 4x^3 - 3x^2 - 2x + 1$ on $-2 \le x \le 0$.

Level 2

116. Find the global extrema of $f(x) = -\frac{4x}{x^2 + 4}$ on $0 < x < \infty$.

117. Find the global extrema of $f(x) = \frac{x^2 - 5}{x + 3}$ on $-\infty < x < -4$.

118. Let $f(x) = \sqrt{2x^2 + 12x + 25}$ where $x \ge 0$.

(a) Prove that $f(x)$ is increasing on $0 \le x < \infty$.

(b) Find the global extrema of $f(x)$ on $0 \le x \le 6$.

119. Let $f(x) = \sqrt{x^3 - 3x^2 + 3x + 8}$ where $x > 0$.

(a) Prove that the curve $y = f(x)$ has no turning points on $x > 0$.

(b) Find the global extrema of $f(x)$ on $x > 0$.

120. For $f(x) = \ln(x^2 - 4x + 5)$ where $0 \le x \le 5$, find its global extrema.

121. Find the global extrema of $f(x) = \dfrac{3\ln x}{x^2}$ on $0 < x < \infty$.

122. Find the global extrema of $f(x) = (2x-1)e^{x-2}$ on $-3 \le x \le 2$.

123. For $f(x) = e^{2x} - 22e^x + 20x$ where $-2 < x < 2$, find its global extrema.

124. Find the global extrema of $f(x) = 2\cos x + x$ on $-\dfrac{\pi}{2} \le x \le \dfrac{\pi}{2}$.

125. Find the global extrema of $f(x) = \tan\dfrac{x}{2} + x$ on $-\dfrac{\pi}{2} \le x \le \dfrac{\pi}{2}$.

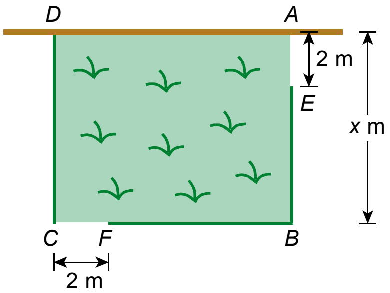
126. Find the global extrema of $f(x) = \sec\dfrac{x}{2} + \tan\dfrac{x}{2}$ on $0 < x < \dfrac{\pi}{3}$.

EXERCISE 8F

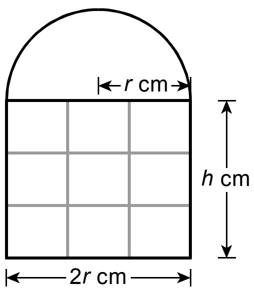
Level 1

127. Find the radius of a sector with the perimeter of 24 cm such that its area is at the maximum.

128. The figure shows a rectangular lawn *ABCD* enclosed by wall *AD* and fences on the remaining sides. *E* and *F* are the points on *AB* and *BC* respectively such that *AE* and *CF* are two entrances to the lawn and *AE* = *CF* = 2 m. It is given that the total length of the fences is 196 m. Let *AB* = *x* m. Find the maximum area of the lawn.



129. In the figure, the window frame is made of iron rods. Upper part of the window frame is a semi-circle with the radius of *r* cm and the lower part is a rectangle with the length of 2*r* cm and width of *h* cm. The rectangular window frame is supported by 2 horizontal iron rods and 2 vertical iron rods.



(a) If the total length of iron rods is 400 cm, express *h* in terms of *r*.

(b) Find the maximum area of the window.

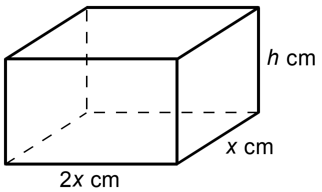
130. At 1 p.m., ship A was 100 km due east of ship B. Ship A sailed due west at a speed of 10 km/h and ship B sailed due north at a speed of 20 km/h. Let s km be the distance between the two ships after t hours ($t \geq 0$) where $t = 0$ corresponds to 1 p.m.

(a) Prove that $s = \sqrt{500t^2 - 2\,000t + 10\,000}$.

(b) When would the two ships be closest to each other? Find the shortest distance between the two ships.

131. If the minimum amount of material is used to make a rectangular box of the capacity 108 cm^3 with a square base but without cover, find the height of the box.

132. In the figure, the length, width and height of a cuboid are $2x$ cm, x cm and h cm respectively. Its surface area is $10\,800\text{ cm}^2$.

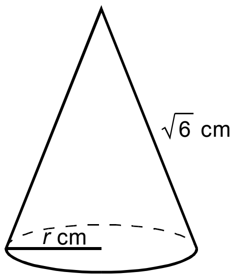


(a) Express h in terms of x.

(b) Express the volume of the cuboid in terms of x.

(c) Find the maximum volume of the cuboid.

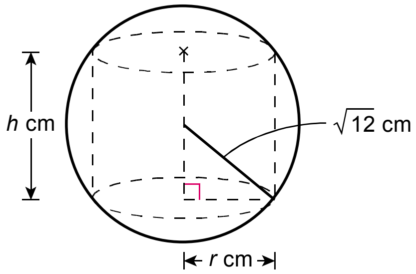
133. The slant height of a right circular cone is $\sqrt{6}$ cm. Let $V\text{ cm}^3$ and r cm be the volume and base radius of the cone respectively.



(a) Express V in terms of r.

(b) Find the base radius of the cone such that its volume is the greatest.

134. As shown in the figure, a person is going to cut out a cylinder from a sphere with the radius of $\sqrt{12}$ cm. Let h cm and r cm be the height and base radius of the cylinder respectively.

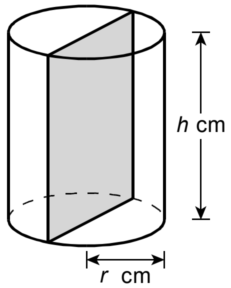


(a) Express r in terms of h.

(b) Let V cm^3 be the volume of the cylinder. Express V in terms of h.

(c) Find the maximum volume of the cylinder as h varies.

135. The figure shows a right cylinder vessel with a lid. Its base radius is r cm, the height is h cm and the capacity is 125 cm^3. The vessel is made of aluminium sheet and its cost is 3 cents per cm^2. There is a steel sheet inside the vessel such that the vessel is divided into two equal parts. The cost of the steel sheet is π cents per cm^2. Let C cents be the production cost of a vessel.



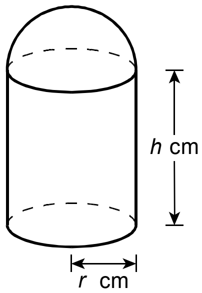
(a) Express h in terms of π and r.

(b) Prove that $C = \dfrac{1\,000}{r} + 6\pi r^2$.

(c) Find the lowest production cost of a vessel. (Give your answer correct to 3 significant figures.)

136. A lead bar with the volume of 576π cm^3 is melted and recast into two spheres. Find the maximum total surface area of the two spheres.

137. The figure shows a right cylindrical container with the base radius of r cm and height of h cm. On the top of it, there is a thin hollow hemispherical lid with the radius of r cm. The capacity of the container is 500 cm^3(excluding the part enclosed with the lid). The container is made of metal sheets. The cost of the lid is p cents per cm^2 where p is a positive constant, the cost of the curved surface of the cylinder is π cents per cm^2 and the cost of the base is 3 cents per cm^2. Let C cents be the production cost of a container.



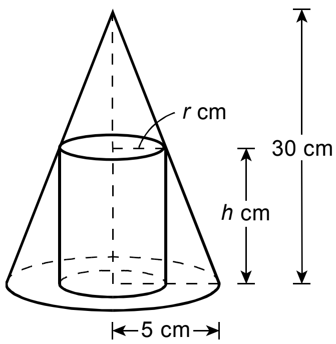
(a) Express h in terms of π and r.

(b) Prove that $C = (2p + 3)\pi r^2 + \dfrac{1\ 000\pi}{r}$.

(c) The production cost of a container is the lowest when $r = 5$. Find the value of p.

(d) Find the lowest production cost of a container.

138. In the figure, there is a right cylinder with the base radius of r cm and height of h cm inside a hollow right circular cone with the base radius of 5 cm and height of 30 cm.

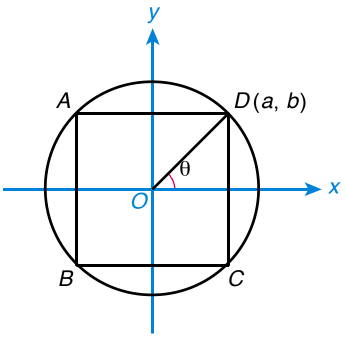


(a) Express h in terms of r.

(b) Find the base radius of the cylinder such that its volume is the greatest.

(c) Find the base radius of the cylinder such that its surface area is the greatest.

139. In the figure, rectangle $ABCD$ is inscribed in a circle with the radius of 3 units, where $0 < \theta < \frac{\pi}{2}$.



(a) Express the coordinates of $D(a, b)$ in terms of θ.

(b) Find the greatest area of $ABCD$.

140. The displacement of a particle with respect to the origin O at time t is given by $x(t) = P\sin t + Q\cos t$, where $0 \le t \le \frac{\pi}{2}$, and both P and Q are constants.

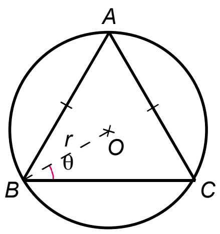
(a) If $x''(t) + x'(t) + 2x(t) = 4\cos t$, find the values of P and Q.

(b) Find the greatest distance between the particle and O.

141. (a) Find the equation of the tangent to the curve $2xy - 1 = 0$ at $(t, \frac{1}{2t})$, where $t > 0$.

(b) If the tangent cuts the x-axis and y-axis at A and B respectively, find the coordinates of the point of contact such that the length of AB is the shortest.

142. An isosceles triangle ABC is inscribed in a circle with the radius of r and the centre at O. $AB = AC$ and $\angle OBC = \theta$, where $0 < \theta < \frac{\pi}{2}$.

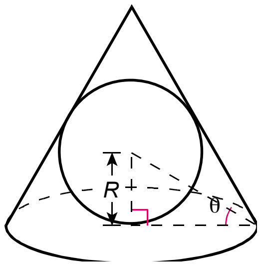


(a) Express the area of ΔABC in terms of r and θ.

(b) Find θ such that the area of ΔABC is at the maximum.

(c) Prove that ΔABC is an equilateral triangle when its area attains the maximum.

143. The figure shows a sphere with the radius of R inscribed in a right circular cone. The line joining the centre of the sphere and any point on the edge of the base of the cone makes an angle θ with the base, where $0° < \theta < 45°$. It is given that ℓ is the slant height of the cone.



(a) Prove that $\ell = R(\cot\theta + \tan 2\theta)$.

(b) Find θ such that the length of the slant height is the shortest. (Give your answer correct to 3 significant figures.)

EXERCISE 8G

Level 1

144. For $y = t^3 - 3t^2 + 4t - 7$, find the rate of change of y with respect to t.

145. For $f(x) = \sqrt{3x^3 - 4x^2 + 5}$, find the rate of change of $f(x)$ with respect to x when $x = 1$.

146. The displacement of an athlete from the starting point t seconds after the start of a 100 m dash is s m, where $s = 0.01t^4 - 0.28t^3 + 2.75t^2$ $(0 \le t \le 11.5)$. Find the velocity and acceleration of the athlete at $t = 5$.

147. The displacement of a person from the starting point after cycling for t hours is s km, where $s = \dfrac{20t^2}{t-6} + 20t$ $(0 \le t \le 3)$. Find the velocity and acceleration of the person after cycling for 1 hour.

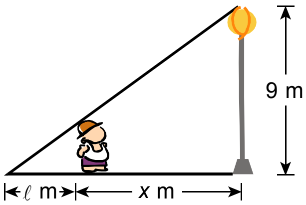
148. A particle moves along the x-axis so that its displacement at any time t (in seconds where $t \ge 0$) is given by $x = 15t^2 - 2t^3$.

(a) Find the velocity and acceleration of the particle at $t = 2$.

(b) Find the total distance travelled by the particle in the first 6 seconds.

149. The horizontal distance between a person and a balloon is 350 m. The balloon is rising vertically at a speed of 120 m/min from the ground. Find the rate of increase of the angle of elevation of the balloon from the person when the balloon is 1 000 m above the ground. (Give your answer correct to 3 significant figures.)

150. In the figure, a man of 1.8 m tall is walking towards a lamp post of 9 m high at a speed of 1.5 m/s. Let x m be the distance between the man and the lamp post, and ℓ m be the length of the shadow of the man.



(a) Express ℓ in terms of x.

(b) Find the rate of change of the length of his shadow with respect to time.

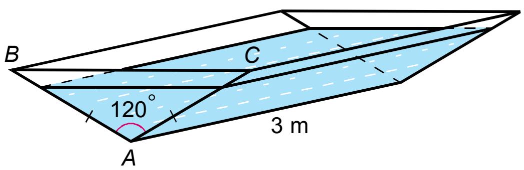
151. At 10 a.m., ship A was 20 km due north of ship B. Ship A sailed due west at 20 km/h and ship B sailed due south at 10 km/h. At 2 p.m., what was the rate of change of the distance between the two ships with respect to time?

152. The volume V cm^3 and surface area A cm^2 of a balloon are given by $V = \frac{4}{3}\pi r^3$ and $A = 4\pi r^2$ respectively, where r cm is the radius of the balloon. The volume of the balloon increases at the rate of 6 cm^3/s. When the volume of the balloon is 36π cm^3, find the rates of change of its radius and surface area with respect to time.

153. The surface area of a cube is diminishing at a rate of 24 cm^2/s. What are the rates of decrease of the length of the side and volume of the cube when the length of its side is 10 cm?

154. The length of a cuboid is increasing at a rate of 1 cm/s, its width is increasing at a rate of 0.5 cm/s and its height is decreasing at a rate of 0.5 cm/s. What is the rate of change of the volume of the cuboid when its length, width and height are 20 cm, 12 cm and 16 cm respectively?

155. The figure shows a container of 3 m long with a uniform cross-section of an isosceles triangle, where $AB = AC$ and $\angle BAC = 120°$. If water is flowing at a rate of 1 m^3/min into the container, find the depth of water when the water level is rising at a rate of $\frac{1}{\sqrt{3}}$ m/min.



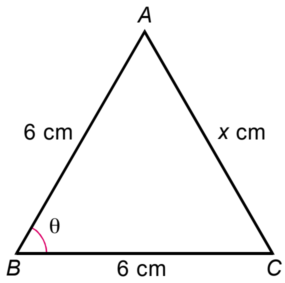
156. The spot of a hill fire was in a circular shape, and its radius was 1 km upon discovery. Thereafter, the area of the spot on fire increased at the constant rate of 3π km^2/h. The firemen arrived 1 hour after the discovery of the hill fire.

(a) Find the radius of the spot (in km) when the firemen arrived.

(b) Find the rate of increase (in km/h) of the radius of the spot when the firemen arrived.

(c) Find the rate of increase (in km/h) of the perimeter of the spot when the firemen arrived.

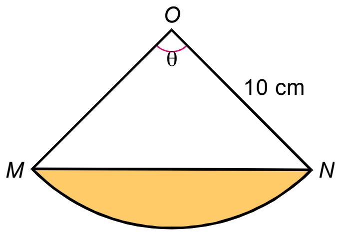
157. In the figure, ΔABC is an isosceles triangle, where $AB = BC = 6$ cm, $AC = x$ cm and $\angle ABC = \theta$. It is known that θ is increasing at a rate of 0.1 rad/s. When $\theta = \frac{\pi}{3}$, find



(a) the rate of change of AC.

(b) the rate of change of the area of ΔABC.

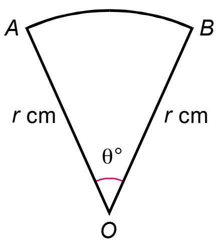
158. In the figure, the radius of sector MON is 10 cm and the angle at the centre is θ.



(a) If θ is increasing at a rate of 0.5 radian/s, find the rate of change of the perimeter of the shaded region when $\theta = \frac{2\pi}{3}$.

(b) If θ is decreasing at a rate of 0.2 radian/s, find the rate of change of the area of the shaded region when $\theta = \frac{5\pi}{6}$.

159. The figure shows a sector with the perimeter of 30 cm. The radius of the sector is r cm and the angle at the centre is $\theta°$.



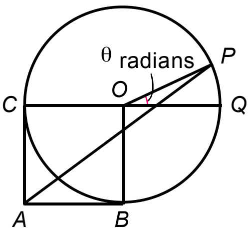
(a) Express θ in terms of r.

(b) Express the area of the sector in terms of r.

(c) If the rate of change of the radius of the sector is -0.1 cm/s, find the rate of change of the area when the radius is 10 cm.

Level 3

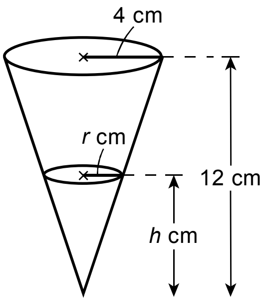
160. In the figure, P is a moving point on a circle with the radius of 1 cm and the centre at O. AP is a rod with the variable length, where A is a fixed point. $ABOC$ is a square, where AB and AC touch the circle at B and C respectively. COQ is a diameter of the circle.



(a) Let $\angle POQ$ be θ radians and the length of AP be ℓ cm. Prove that $\ell^2 = 3 + 2(\sin\theta + \cos\theta)$.

(b) If P rotates at a rate of 3 radians per second, find the rate of change of the length of AP when $\theta = \frac{\pi}{3}$. (Give your answer correct to 2 decimal places.)

161. The base radius and height of an inverted right circular cone are 4 cm and 12 cm respectively. Water is flowing out at a constant rate of $\frac{\pi}{2}$ cm^3/s from the vertex of the cone.

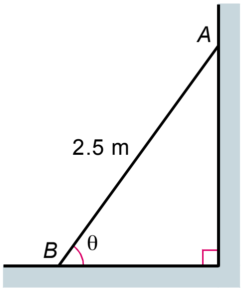


(a) Show that the volume of water in the cone V (in cm^3) can be expressed as $V = \frac{1}{27}\pi h^3$ when the depth of water is h cm.

(b) When the depth of water is 6 cm, find the rate of change of the depth of water with respect to time.

(c) When the depth of water is 6 cm, find the rate of change of the area of the water surface with respect to time.

162. In the figure, a ladder with the length of 2.5 m leans against a wall. The point of contact A between the ladder and the wall slides down at a speed of 0.4 m / s. When A is 2.4 m above the ground,



(a) find the sliding speed of B away from the wall.

(b) find the rate of change of the angle θ between the ladder and the ground.

(c) find the rate of change of the slope of the ladder.

(Give your answers correct to 3 significant figures.)