

Limits and the Number e

E XERCISE 6A*Level 1*

1. Find $\lim_{x \rightarrow 5} (-\sqrt{3})$.
2. Find $\lim_{x \rightarrow -2} (x^3 + 2x^2 + 4)$.
3. Find $\lim_{x \rightarrow 2} (x - 2)(x^2 + 8)$.
4. Find $\lim_{x \rightarrow 4} \frac{x^2 + 16}{x + 4}$.
5. Find $\lim_{x \rightarrow -1} \frac{x^2 + 3x + 4}{x + 2}$.
6. Find $\lim_{x \rightarrow 5} \frac{x^2 - 25}{x - 5}$.
7. Find $\lim_{x \rightarrow -3} \frac{x + 3}{x^2 + 5x + 6}$.
8. Find $\lim_{x \rightarrow 6} \frac{x^2 - 2x - 24}{(x - 2)(x - 6)}$.
9. Find $\lim_{x \rightarrow 8} \frac{(x + 7)(8 - x)}{x^2 - 7x - 8}$.
10. Find $\lim_{x \rightarrow -2} \frac{x^2 - 2x - 8}{x^2 + 7x + 10}$.
11. Find $\lim_{x \rightarrow \sqrt{3}} \frac{12 - 7x^2 + x^4}{18 - 3x^2 - x^4}$.
12. Find $\lim_{x \rightarrow 0} \frac{\frac{1}{x+5} - \frac{1}{5}}{x}$.
13. Find $\lim_{x \rightarrow 0} \left[\frac{1}{2x} - \frac{1}{x(x+2)} \right]$.
14. Find $\lim_{x \rightarrow -4} \frac{\frac{2}{x} + \frac{1}{2}}{x^2 - 16}$.

15. Find $\lim_{x \rightarrow -4} \frac{x+4}{x^3+64}$.
16. Find $\lim_{x \rightarrow 5} \frac{x^3-125}{25-x^2}$.
17. Find $\lim_{x \rightarrow -3} \frac{x^2-3x-18}{x^3+27}$.
18. Find $\lim_{x \rightarrow 2} \frac{4^x-16}{2^x-4}$.

Level 2

19. Find $\lim_{h \rightarrow 0} \frac{(x-2h)^3 - x^3}{h}$.
20. Find $\lim_{h \rightarrow -2} \frac{(x-2h^2)^2 - (x+4h)^2}{h+2}$.
21. Find $\lim_{x \rightarrow 3} \frac{x^2-7x+12}{x^2-6x+9}$.
22. Find $\lim_{x \rightarrow -\frac{3h}{2}} \frac{2x^2+5xh+3h^2}{4x^2+12xh+9h^2}$.
23. Find $\lim_{x \rightarrow 4} \frac{x^2-16}{\sqrt{x}-4}$.
24. Find $\lim_{x \rightarrow -5} \frac{\sqrt{x+5}}{x^2-25}$.
25. Find $\lim_{x \rightarrow 16} \frac{16-x}{4-\sqrt{x}}$.
26. Find $\lim_{x \rightarrow -\sqrt{2}} \frac{x+\sqrt{2}}{x^4-4}$.
27. Find $\lim_{x \rightarrow 0} \frac{\sqrt{9+x}-3}{\sqrt{1+x}-1}$.
28. Find $\lim_{x \rightarrow -2} \frac{x^2+6x+8}{2-\sqrt{x^3+12}}$.
29. Find $\lim_{x \rightarrow 3} \frac{\sqrt{3x+7}-4}{3-\sqrt{x+6}}$.
30. If $\lim_{x \rightarrow 0} \frac{3x^3+bx}{4x^2-3x} = 2$, where b is a constant, find the value of b .

Level 3

31. Find $\lim_{x \rightarrow 5} \frac{\sqrt{x^2 + 5x} - \sqrt{50}}{\sqrt{x^2 - 10} - \sqrt{15}}$.
32. Find $\lim_{h \rightarrow 0} \frac{(x-h)\sqrt{x+h} - x\sqrt{x}}{\sqrt{x-h} - \sqrt{x}}$.
33. Find $\left[\lim_{x \rightarrow h} \frac{(x+h)^3 + (x-3h)^3}{x-h} \right] \cdot \left[\lim_{x \rightarrow 2h} \frac{\frac{1}{x+h} - \frac{1}{4x-5h}}{x-2h} \right]$.
34. If $\lim_{x \rightarrow a} \frac{x^2 - 25}{\sqrt{x^2 - 9} - 4} = 8$, where a is a constant, find the value of a .

EXERCISE 6B

Level 1

35. Find $\lim_{x \rightarrow \infty} \frac{40}{5x + 8}$.
36. Find $\lim_{x \rightarrow -\infty} \frac{100}{8x - 9x^5}$.
37. Find $\lim_{x \rightarrow \infty} \frac{4x + 3}{8x - 3}$.
38. Find $\lim_{x \rightarrow -\infty} \frac{x^3 + 3x}{2x^2 - 4x^3}$.
39. Find $\lim_{x \rightarrow \infty} \frac{x^3 + 4x^2 + 5x + 7}{(x^2 + 3)(x^2 - 9)}$.
40. Find $\lim_{x \rightarrow \infty} \frac{(x+3)(x+6)(x^2-5)}{(x^2-x^3)(2x^2-1)}$.
41. Find $\lim_{x \rightarrow \infty} \frac{5x^3 - 5x + 6}{50x^2 + 27x + 4\,000}$.
42. Find $\lim_{x \rightarrow \infty} \frac{(3x-x^2)(x^3+5)}{(x+1)(x+2)(x+4)(x+8)}$.
43. Find $\lim_{x \rightarrow -\infty} \frac{(x+1)(x^2-2)}{2x^2-3x+4}$.
44. Find $\lim_{x \rightarrow \infty} \left(\frac{6x-5}{3x+4} \right)^6$.

45. Find $\lim_{x \rightarrow \infty} \sqrt{\frac{6x^2 + 8x}{24x^2 - 25}}$.
46. Find $\lim_{x \rightarrow -\infty} \sqrt{\frac{(x^2 + 2)(x + 5)}{4x^3 + 6x + 9}}$.
47. Find $\lim_{x \rightarrow -\infty} \sqrt{\frac{x^2 - 4x + 1}{3x^3 - 5x^2 - 2}}$.
48. Find $\lim_{x \rightarrow -\infty} \sqrt{\frac{3 - 4x - 5x^2}{5 - 2x^3 + 9x}}$.
49. Find $\lim_{x \rightarrow \infty} \sqrt[3]{\frac{3 + 5x - 16x^3}{2x^3 + 8}}$.
50. Find $\lim_{x \rightarrow \infty} \frac{\sqrt{4x^8 + 2x^4 + 1}}{6x^4 - 3x^2 - 9}$.
51. Find $\lim_{x \rightarrow \infty} \frac{4x^2 - 2}{\sqrt{3x^3 - 2x^2 - x + 5}}$.
52. Find $\lim_{x \rightarrow -\infty} (5^x - \sqrt{5^{2x} - 5^x + 9})$.

Level 2

53. Find $\lim_{x \rightarrow -\infty} \frac{9^x - 3^{-x}}{9^x + 3^{-x}}$.
54. Find $\lim_{x \rightarrow \infty} \frac{2^x - 6}{2^{x+3} + 9}$.
55. Find $\lim_{x \rightarrow -\infty} \frac{8x}{\sqrt{4x^2 - 6}}$.
56. Find $\lim_{x \rightarrow -\infty} \frac{\sqrt{4x^4 - 9}}{8x^2 + 6x + 7}$.
57. Find $\lim_{x \rightarrow -\infty} \frac{\sqrt{9x^2 + 6} - 3}{x}$.
58. Find $\lim_{x \rightarrow -\infty} \frac{x + 1}{\sqrt{4x^2 + 5} - 2}$.
59. Find $\lim_{x \rightarrow \infty} (\sqrt{x^2 - 2x - 3} - x)$.

60. Find $\lim_{x \rightarrow -\infty} (\sqrt{9x^2 - 10x + 11} + 3x)$.
61. Find $\lim_{x \rightarrow \infty} (2^x - \sqrt{4^x - 2^x + 9})$.
62. Find $\lim_{x \rightarrow \infty} \frac{1}{\sqrt{9x^2 - 4x} - 3x + 1}$.
63. Find $\lim_{x \rightarrow \infty} [(\sqrt{x+9} - \sqrt{x-5})\sqrt{x}]$.
64. Find $\lim_{x \rightarrow -\infty} (\sqrt{2x^2 - 3x + 5} - \sqrt{2x^2 + 3x - 5})$.
65. Find $\lim_{x \rightarrow \infty} (\sqrt{\frac{9x^3}{4x+3}} - \frac{3x}{2})$.
66. Let $f(x) = \frac{hx^2 + kx}{2x^2 - x}$, where h and k are constants. It is given that $\lim_{x \rightarrow 0} f(x) = 8$ and $\lim_{x \rightarrow \infty} f(x) = 2$.
Find the values of h and k .
67. (a) Prove, by mathematical induction, that for all positive integers n ,
- $$1^2 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}.$$
- (b) Hence find $\lim_{n \rightarrow \infty} \frac{1^3 + 2^3 + 3^3 + \cdots + n^3}{1^3 + 2^3 + 3^3 + \cdots + (2n)^3}$.

Level 3

68. Let $f(x) = \frac{\sqrt{hx^2 - 6x}}{2x} + \sqrt{\frac{kx^2 - 16x + 25}{3x^2}}$, where h and k are constants. It is given that $\lim_{x \rightarrow \infty} f(x) = 5$
and $\lim_{x \rightarrow -\infty} f(x) = 1$, and both $\lim_{x \rightarrow \pm\infty} \frac{\sqrt{hx^2 - 6x}}{2x}$ and $\lim_{x \rightarrow \pm\infty} \sqrt{\frac{kx^2 - 16x + 25}{3x^2}}$ exist. Find the values of h and k .
69. Find $\lim_{x \rightarrow -\infty} (2x + \sqrt{\frac{4x^3 - 3}{x+1}})$.

E XERCISE 6C*Level 1*

70. Find $\lim_{x \rightarrow 0} \frac{3x}{\sin 2x}$.
71. Find $\lim_{x \rightarrow 0} 2x \cot \frac{x}{2}$.
72. Find $\lim_{x \rightarrow 0} \sin 2x \csc 4x$.
73. Find $\lim_{x \rightarrow 0} \frac{x \tan 2x}{\sin^2 4x}$.
74. Find $\lim_{x \rightarrow 0} \frac{\sin 2x^2}{\sin^2 2x}$.
75. Find $\lim_{x \rightarrow 0} \frac{\sec^2 x - 1}{x \tan 3x}$.
76. Find $\lim_{x \rightarrow 0} \frac{4x - \sin 2x}{\sin \frac{x}{2} + 6x^2}$.
77. Find $\lim_{x \rightarrow 0} \frac{1 - \cos \frac{x}{2}}{x^2}$.
78. Find $\lim_{x \rightarrow 0} \frac{\cos 2x - \cos x}{3x}$.
79. Find $\lim_{x \rightarrow 0} \frac{\sin 3x - \sin 7x}{x \cos^2 x}$.
80. Find $\lim_{x \rightarrow 0} \frac{x \tan 5x}{\cos 4x - \cos 2x}$.
81. Find $\lim_{x \rightarrow 0} \frac{4x^2 + \cos 6x - \cos 2x}{3x(\sin 5x + \sin x)}$.
82. Find $\lim_{x \rightarrow \alpha} \frac{x - \alpha}{\cos 2x - \cos 2\alpha}$.
83. Find $\lim_{x \rightarrow \beta} \frac{\tan x - \tan \beta}{x - \beta}$.
84. Find $\lim_{x \rightarrow 0} \frac{\sin x}{\sqrt{2} \sin(2x + \frac{\pi}{4}) - 1}$.
85. Find $\lim_{x \rightarrow 0} \frac{\sqrt{3} \tan(3x - \frac{\pi}{6}) + 1}{x}$.
86. Find $\lim_{x \rightarrow 0} \frac{x \sin x \tan x}{\sqrt{1 - \tan 2x} - \sqrt{1 - \sin 2x}}$.

87. Find $\lim_{x \rightarrow \frac{\pi}{6}} \frac{\sin(3x - \frac{\pi}{2})}{1 - 2\cos 2x}$.
88. Find $\lim_{x \rightarrow \frac{\pi}{2}} \frac{(x - \frac{\pi}{2})\cos x}{1 + \sin 3x}$.
89. Find $\lim_{x \rightarrow \frac{2\pi}{3}} \frac{\cos(x + \frac{\pi}{3}) + 1}{(x - \frac{2\pi}{3})\sin(x - \frac{2\pi}{3})}$.
90. Find $\lim_{x \rightarrow \frac{\pi}{4}} \frac{[\sin(x - \frac{\pi}{4}) - \sin(3x - \frac{3\pi}{4})]^2}{\sin 4x \sin(\frac{3\pi}{2} - 2x)}$.

EXERCISE 6D

Level 1

91. Find $\lim_{x \rightarrow 0} e^x \ln(2x^2 + 1)$.
92. Find $\lim_{x \rightarrow 2} \ln(e^{x^2-3} + 2x - x^2)$.
93. Find $\lim_{x \rightarrow \infty} \frac{8e^x - 36e^{2x}}{e^{3x}}$.
94. Find $\lim_{x \rightarrow -\infty} \frac{e^{2x} + 8e^x - 3x}{2x}$.
95. Find $\lim_{x \rightarrow \infty} (1 + \frac{1}{5x})^x$.
96. Find $\lim_{x \rightarrow \infty} (1 - \frac{3}{x})^{x+2}$.
97. Find $\lim_{x \rightarrow 0} (1 + \frac{x}{2})^{\frac{1}{x}}$.
98. Find $\lim_{x \rightarrow 0} (1 - 3x^2)^{\frac{1}{6x^2}}$.
99. Find $\lim_{x \rightarrow \infty} (1 - \frac{1}{3\sqrt{x}})^{\sqrt{x}}$.
100. Find $\lim_{x \rightarrow \infty} (1 + \frac{4}{3x})^{2x+4}$.
101. Find $\lim_{x \rightarrow \infty} (\frac{3x-2}{3x+4})^x$.

102. Find $\lim_{x \rightarrow \infty} \left(\frac{2x^2 + 3x}{2x^2 - 5x} \right)^{\frac{x}{2}}$.

103. Find $\lim_{x \rightarrow \infty} \left(\frac{3}{3 + \frac{4}{x-2}} \right)^{-x+2}$.

104. Find $\lim_{x \rightarrow 0} \frac{e^{x+4} - e^4}{4x}$.

105. Find $\lim_{x \rightarrow 0} \frac{e^3 - e^{2x+3}}{6x}$.

106. Find $\lim_{x \rightarrow 0} \frac{1 - e^{-6x}}{2x}$.

107. Find $\lim_{x \rightarrow \infty} 2x \ln\left(1 + \frac{2}{x}\right)$.

Level 2

108. Find $\lim_{x \rightarrow 0} \frac{\ln(2x+1)}{3x}$.

109. Find $\lim_{x \rightarrow 0} \frac{e^x - e^{-x}}{3xe^x}$.

110. Find $\lim_{x \rightarrow 0} \frac{e^{5x} - e^{3x}}{2x}$.

111. Find $\lim_{x \rightarrow 0} \frac{1 - e^{-x^2}}{2x}$.

112. Find $\lim_{x \rightarrow 0} \frac{e^{x^4} - e^{x^2}}{2x^2}$.

113. Find $\lim_{x \rightarrow 0} \frac{e^x - 2e^{-x} + 1}{x}$.

114. Find $\lim_{x \rightarrow -\infty} \left(\frac{2x^2 + 5}{2x^2 + 7} \right)^{x^2}$.

115. Find $\lim_{x \rightarrow \infty} \left(\frac{x^2 + 6x + 8}{x^2 + 7x - 8} \right)^x$.

116. Find $\lim_{x \rightarrow \infty} \left(\frac{x^2 + 3x - 18}{x^2 - 25} \right)^x$.

117. Find $\lim_{x \rightarrow 0} \left(\frac{18x^2 + 9x + 1}{4x^2 + 5x + 1} \right)^{\frac{1}{x}}$.

118. Find $\lim_{x \rightarrow 0} \left(\frac{6 + \frac{5}{x} + \frac{1}{x^2}}{30 - \frac{11}{x} + \frac{1}{x^2}} \right)^{\frac{1}{x}}$.

119. (a) Express $1 + \frac{5}{x} + \frac{6}{x^2}$ in the form of $(1 + \frac{a}{x})(1 + \frac{b}{x})$, where a and b are constants.

(b) Hence find $\lim_{x \rightarrow \infty} (1 + \frac{5}{x} + \frac{6}{x^2})^x$.

120. Find $\lim_{x \rightarrow \infty} x \ln(1 + \frac{1}{x} - \frac{2}{x^2})$.

Level 3

121. Find $\lim_{x \rightarrow 0} \frac{\sin(1 - e^{2x})}{1 - e^x}$.

122. Find $\lim_{x \rightarrow 0} \frac{e^{2x} + e^{-2x} - 2}{x \sin 4x}$.

123. Solve the equation $e^{2x} - 6e^x + 5 = 0$.

124. Solve the equation $e^x - 8e^{-x} + 2 = 0$.

125. Solve the equation $2\ln(x+3) - \ln(x+5) = 0$.

126. Solve the equation $[\ln(x+2)]^2 = 3\ln(x+2)$.