Chapter 15

# Multiplications of Vectors

## Exercise 15A

*Level 1*

1. Find the value of $\mathbf{a}\cdot\mathbf{b}$ for each of the following, where $\theta$ is the angle between $\mathbf{a}$ and $\mathbf{b}$.

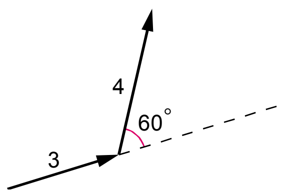
**(a)** $|\mathbf{a}| = 8$, $|\mathbf{b}| = 4$, $\theta = 60°$ **(b)** $|\mathbf{a}| = 3$, $|\mathbf{b}| = 7$, $\theta = 45°$

2. Find the value of $\mathbf{a}\cdot\mathbf{b}$ for each of the following, where $\theta$ is the angle between $\mathbf{a}$ and $\mathbf{b}$.

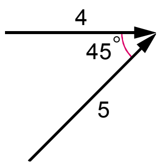
**(a)** $|\mathbf{a}| = 9$, $|\mathbf{b}| = 2$, $\theta = 90°$ **(b)** $|\mathbf{a}| = 2$, $|\mathbf{b}| = 4$, $\theta = 150°$

3. Find the value of the scalar product for each of the following sets of vectors.

**(a)**

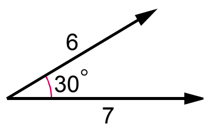



**(b)**

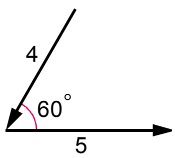



4. Find the value of the scalar product for each of the following sets of vectors.

**(a)**




**(b)**




5. Find the value of each of the following scalar products.

**(a)** $\mathbf{j}\cdot(-\mathbf{k})$ **(b)** $(2\mathbf{i}-\mathbf{j})\cdot(\mathbf{i}+3\mathbf{j})$

6. Find the value of each of the following scalar products.

**(a)** $(-\mathbf{i}-4\mathbf{j})\cdot(2\mathbf{i}+3\mathbf{j})$ **(b)** $(\mathbf{i}-2\mathbf{j}-3\mathbf{k})\cdot(-\mathbf{i}+\mathbf{k})$

**(*) Out syl after HKDSE 2022**
**Scalar Triple Products**

$(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$

$(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} = (\mathbf{b}\times\mathbf{c})\cdot\mathbf{a} = (\mathbf{c}\times\mathbf{a})\cdot\mathbf{b}$

**volume of a parallelepiped**

**7.** Determine whether each of the following sets of vectors are orthogonal vectors.

**(a)** $\mathbf{i} - 4\mathbf{j}$, $-4\mathbf{i} + \mathbf{j}$

**(b)** $2\mathbf{i} + 3\mathbf{j} - 5\mathbf{k}$, $-2\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$

**8.** Determine whether each of the following sets of vectors are orthogonal vectors.

**(a)** $-2\mathbf{i} + 3\mathbf{j}$, $6\mathbf{i} + 4\mathbf{j}$

**(b)** $\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$, $3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$

**9.** If $\mathbf{a} = \mathbf{i} - \mathbf{j} + \mathbf{k}$ and $\mathbf{b} = -\mathbf{i} - \mathbf{j} + 2\mathbf{k}$, find the angle between $\mathbf{a}$ and $\mathbf{b}$. (Give your answer correct to the nearest 0.1°.)

**10.** If $\mathbf{a} = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$ and $\mathbf{b} = -\mathbf{i} + \mathbf{j} + \mathbf{k}$, find the angle between $\mathbf{a}$ and $\mathbf{b}$. (Give your answer correct to the nearest 0.1°.)

**11.** It is given that $\mathbf{u} = 7\mathbf{i} + \mathbf{j}$ and $\mathbf{v} = \mathbf{i} - \mathbf{j}$.

**(a)** Find the value of $\mathbf{u} \cdot \mathbf{v}$.

**(b)** Find the angle between $\mathbf{u}$ and $\mathbf{v}$. (Give your answer correct to the nearest 0.1°.)

**12.** It is given that $\mathbf{u} = 2\mathbf{i} - \mathbf{j} - \mathbf{k}$ and $\mathbf{v} = \mathbf{i} + 3\mathbf{j} - \mathbf{k}$.

**(a)** Find the value of $\mathbf{u} \cdot \mathbf{v}$.

**(b)** Find the angle between $\mathbf{u}$ and $\mathbf{v}$.

**13.** It is given that $|\mathbf{a}| = 2$, $|\mathbf{b}| = \sqrt{3}$ and the angle between $\mathbf{a}$ and $\mathbf{b}$ is $\frac{\pi}{6}$. If $\mathbf{p} = \mathbf{a} + \mathbf{b}$ and $\mathbf{q} = \mathbf{a} - \mathbf{b}$, find the angle between $\mathbf{p}$ and $\mathbf{q}$. (Give your answer correct to the nearest 0.1°.)

**14.** $\mathbf{a}$ and $\mathbf{b}$ are orthogonal vectors, where $|\mathbf{a}| = 4$ and $|\mathbf{b}| = 2$. If $\mathbf{u} = u_1\mathbf{a} + u_2\mathbf{b}$ and $\mathbf{v} = v_1\mathbf{a} + v_2\mathbf{b}$ are also orthogonal vectors, prove that $4u_1v_1 + u_2v_2 = 0$.

**15.** If $\mathbf{a}$ and $\mathbf{b}$ are two non-zero vectors and they are not parallel to each other, find the condition such that $\mathbf{a} + \mathbf{b}$ is perpendicular to $\mathbf{a} - \mathbf{b}$.

**16.** It is given that $\overrightarrow{AB} = 2\mathbf{i} + 4\mathbf{j}$ and $\overrightarrow{AC} = 6\mathbf{i} - 2\mathbf{j}$, $D$ is the mid-point of $BC$, find the value of $\overrightarrow{AD} \cdot \overrightarrow{BC}$.

**17.** It is given that $\overrightarrow{OA} = \mathbf{i} + 2\mathbf{j}$ and $\overrightarrow{OB} = 6\mathbf{i} + 3\mathbf{j}$. If $OB \perp AC$ and $\overrightarrow{OC} = k\overrightarrow{OB}$,

**(a)** express $\overrightarrow{AC}$ in terms of $k$, $\mathbf{i}$ and $\mathbf{j}$.

**(b)** find the value of $k$.

**18.** It is given that $\overrightarrow{OA} = 6\mathbf{i} - 3\mathbf{j}$ and $\overrightarrow{OB} = 2\mathbf{i} - 6\mathbf{j}$. If $OA \perp BC$ and $\overrightarrow{OC} = k\overrightarrow{OA}$,

**(a)** express $\overrightarrow{BC}$ in terms of $k$, $\mathbf{i}$ and $\mathbf{j}$.

**(b)** find the value of $k$.

**(c)** Hence find the coordinates of $C$.

**19.** It is given that $\left|\overrightarrow{OA}\right| = 2$, $\left|\overrightarrow{OB}\right| = 1$ and $\angle AOB = 60°$.

**(a)** Find the value of $\overrightarrow{OA} \cdot \overrightarrow{OB}$.

**(b)** Find the value of $\left|\overrightarrow{OA} - 2\overrightarrow{OB}\right|$.

**20.** It is given that $\left|\overrightarrow{OA}\right| = 4$, $\left|\overrightarrow{OB}\right| = \sqrt{3}$ and $\angle AOB = \frac{5\pi}{6}$.

**(a)** Find the value of $\overrightarrow{OA} \cdot \overrightarrow{OB}$.

**(b)** Find the value of $\left|\overrightarrow{OA} + 3\overrightarrow{OB}\right|$.

**21.** If $|\mathbf{a}| = 4$, $|\mathbf{b}| = 5$ and the angle between $\mathbf{a}$ and $\mathbf{b}$ is $60°$, find the value of $|\mathbf{a} - \mathbf{b}|$.

**22.** $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are three vectors, where $|\mathbf{a}| = 3$, $|\mathbf{b}| = 6$, $|\mathbf{c}| = 7$ and $|\mathbf{a} + \mathbf{b} + \mathbf{c}| = 0$. Find the value of $\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{c} + \mathbf{c} \cdot \mathbf{a}$.

### *Level 2*

**23.** $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{p}$ are vectors, where $\mathbf{a} \cdot \mathbf{c} = 4$, $\mathbf{b} \cdot \mathbf{c} = -1$ and $\mathbf{p} = k\mathbf{a} + 2\mathbf{b}$. If $\mathbf{p}$ is perpendicular to $\mathbf{c}$, find the value of $k$.

**24.** It is given that $\mathbf{a} = \mathbf{i} + \mathbf{j}$, $\mathbf{b} = \mathbf{j} + \mathbf{k}$ and $\mathbf{c} = \mathbf{i} + m\mathbf{j} + n\mathbf{k}$, where $m$ and $n$ are constants. If $\mathbf{c}$ is perpendicular to $\mathbf{a}$ and $\mathbf{b}$, find the values of $m$ and $n$.

**25.** $\mathbf{a}$ and $\mathbf{b}$ are two vectors, where $|\mathbf{a}| = 5$, $|\mathbf{b}| = 2$ and the angle between $\mathbf{a}$ and $\mathbf{b}$ is $120°$.

**(a)** Find the value of $\mathbf{a} \cdot \mathbf{b}$.

**(b)** If $\mathbf{a} + k\mathbf{b}$ and $\mathbf{a} + 3\mathbf{b}$ are perpendicular to each other, find the value of $k$.

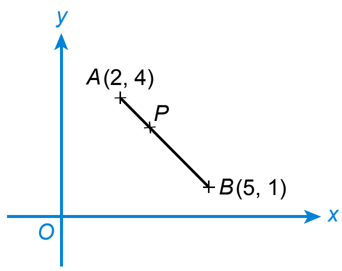
**26.** $\mathbf{p}$ and $\mathbf{q}$ are two vectors, where $\mathbf{p} = 5\mathbf{i} + 12\mathbf{j}$, $|\mathbf{q}| = \sqrt{3}$ and the angle between $\mathbf{p}$ and $\mathbf{q}$ is $150°$.

**(a)** Find the value of $|\mathbf{p}|$.

**(b)** Find the value of $\mathbf{p} \cdot \mathbf{q}$.

**(c)** If $k\mathbf{p} + \mathbf{q}$ and $\mathbf{q}$ are perpendicular to each other, find the value of $k$.

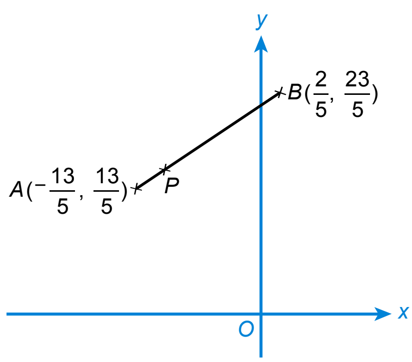
**27.** In the figure, the coordinates of $A$ and $B$ are (2, 4) and (5, 1) respectively. $P$ is a point on $AB$ such that $AP : PB = 1 : r$.




**(a)** Express $\overrightarrow{OP}$ in terms of $r$, $\mathbf{i}$ and $\mathbf{j}$.

**(b)** Hence, if $OP \perp AB$, find the value of $r$.

**28.** In the figure, the coordinates of $A$ and $B$ are $(-\frac{13}{5}, \frac{13}{5})$ and $(\frac{2}{5}, \frac{23}{5})$ respectively. $P$ is a point on $AB$ such that $AP : PB = 1 : r$.




**(a)** Express $\overrightarrow{OP}$ in terms of $r$, $\mathbf{i}$ and $\mathbf{j}$.

**(b)** Hence, if $OP \perp AB$, find the coordinates of $P$.

**(c)** Find the shortest distance from $O$ to $AB$.

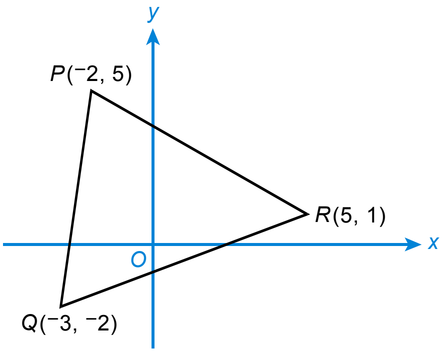
**29.** The coordinates of $A$ and $B$ are $(2u, -3u, u+1)$ and $(2v-1, v-2, -v-1)$ respectively. If $\overrightarrow{AB}$ is perpendicular to $\mathbf{j}+\mathbf{k}$ and $-7\mathbf{i}-\mathbf{j}-2\mathbf{k}$, find the values of $u$ and $v$.

**30.** The coordinates of $A$ and $B$ are $(2u+1, -u, u+1)$ and $(2v, v, v-1)$ respectively. If $\overrightarrow{AB}$ is perpendicular to $3\mathbf{i}-\mathbf{j}$ and $-3\mathbf{i}+2\mathbf{j}+3\mathbf{k}$, find the coordinates of $A$ and $B$.

**31.** Let $\mathbf{a}$ and $\mathbf{b}$ be two vectors. If $|3\mathbf{a}-4\mathbf{b}|=5$ and $|3\mathbf{a}+4\mathbf{b}|=11$, find the value of $\mathbf{a}\cdot\mathbf{b}$.

**32.** Let $\mathbf{a}$ and $\mathbf{b}$ be two vectors. If $|2\mathbf{a}-5\mathbf{b}|=3$ and $|2\mathbf{a}+5\mathbf{b}|=1$, find the value of $\mathbf{a}\cdot\mathbf{b}$.

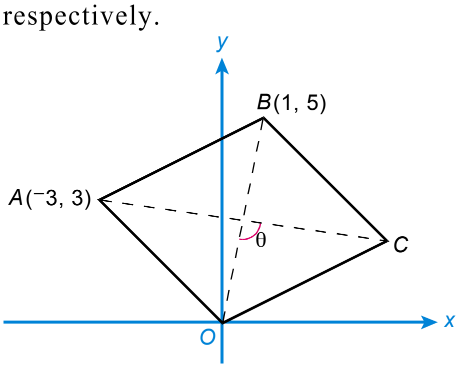
**33.** In the figure, the position vectors of $P$, $Q$ and $R$ are $\overrightarrow{OP}=-2\mathbf{i}+5\mathbf{j}$, $\overrightarrow{OQ}=-3\mathbf{i}-2\mathbf{j}$ and $\overrightarrow{OR}=5\mathbf{i}+\mathbf{j}$ respectively.




**(a)** Find $\overrightarrow{PQ}$ and $\overrightarrow{PR}$.

**(b)** Consider $\overrightarrow{PQ}\cdot\overrightarrow{PR}$, find $\angle QPR$. (Give your answer correct to the nearest degree.)

**34.** The figure shows parallelogram $OABC$. The position vectors of $A$ and $B$ are $-3\mathbf{i}+3\mathbf{j}$ and $\mathbf{i}+5\mathbf{j}$ respectively.




**(a)** Find $\overrightarrow{OC}$ and $\overrightarrow{AC}$.

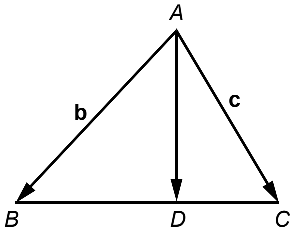
**(b)** Find the obtuse angle $\theta$ between $OB$ and $AC$. (Give your answer correct to the nearest degree.)

**35.** In $\Delta ABC$, $\overrightarrow{AB} = 12\mathbf{i} + 5\mathbf{j}$ and $\overrightarrow{AC} = 3\mathbf{i} - 4\mathbf{j}$.

**(a)** Find $\cos\angle CAB$.

**(b)** Find the area of $\Delta ABC$.

**36.** In $\Delta ABC$, $\overrightarrow{AB} = \mathbf{b}$ and $\overrightarrow{AC} = \mathbf{c}$.



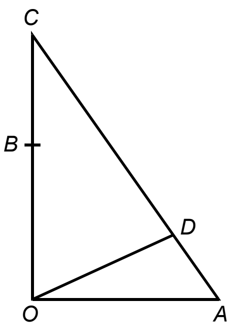

**(a)** $D$ is a point on $BC$ such that $BD : DC = 1 : t$. Express $\overrightarrow{AD}$ and $\overrightarrow{BC}$ in terms of **b**, **c** and $t$.

**(b)** It is given that $AD$ and $BC$ are perpendicular to each other, prove that $t = -\dfrac{\mathbf{c}\cdot(\mathbf{b}-\mathbf{c})}{\mathbf{b}\cdot(\mathbf{b}-\mathbf{c})}$.

**(c)** Hence prove that $\overrightarrow{AD} = [\dfrac{\mathbf{b}\cdot(\mathbf{b}-\mathbf{c})}{(\mathbf{b}-\mathbf{c})\cdot(\mathbf{b}-\mathbf{c})}]\mathbf{c} - [\dfrac{\mathbf{c}\cdot(\mathbf{b}-\mathbf{c})}{(\mathbf{b}-\mathbf{c})\cdot(\mathbf{b}-\mathbf{c})}]\mathbf{b}$.

**37.** In $\Delta ABC$, $\overrightarrow{AB} = 7\mathbf{i} + 4\mathbf{j}$, $\overrightarrow{AC} = -2\mathbf{i} + 8\mathbf{j}$, $P$ and $Q$ are points on $BC$ such that $BP : PQ : QC = 1 : 2 : 1$. Find $\angle APQ$. (Give your answer correct to 3 significant figures.)

**38.** In the figure, $\overrightarrow{OA} = \mathbf{i}$ and $\overrightarrow{OB} = \mathbf{j}$. $C$ is a point on $OB$ produced such that $BC = k$, where $k > 0$. $D$ is a point on $AC$ such that $AD : DC = 1 : 3$.




**(a)** Prove that $\overrightarrow{OD} = \dfrac{3}{4}\mathbf{i} + \dfrac{1+k}{4}\mathbf{j}$.

**(b)** If $\overrightarrow{OD}$ is perpendicular to $\overrightarrow{AC}$, find

**(i)** the value of $k$.

**(ii)** the value of $\left|\overrightarrow{OD}\right|$.

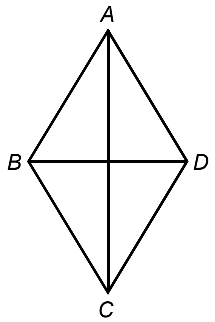
**39.** In $\Delta ABC$, $\overrightarrow{AB} = \mathbf{a}$ and $\overrightarrow{AC} = \mathbf{b}$. It is given that $|\mathbf{a}| = 4$, $|\mathbf{b}| = 3$ and $(2\mathbf{a} - 3\mathbf{b}) \cdot (2\mathbf{a} + \mathbf{b}) = 61$.

**(a)** Find the angle $\theta$ between $\mathbf{a}$ and $\mathbf{b}$.

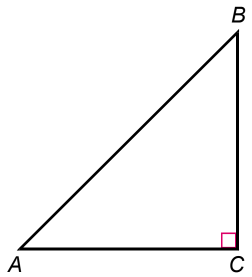
**(b)** Find the values of $|\mathbf{a} + \mathbf{b}|$ and $|\mathbf{a} - \mathbf{b}|$.

**(c)** Find the area of $\Delta ABC$.

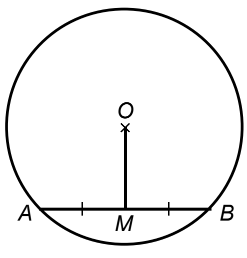
**40.** In the figure, $ABCD$ is a rhombus. Prove that the diagonals of a rhombus are perpendicular to each other by using vectors.



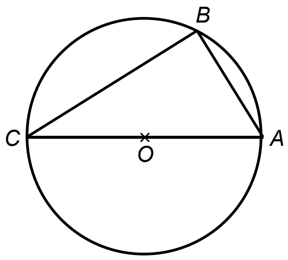

**41.** In the figure, $ABC$ is a right-angled triangle, where $\angle C = 90°$. Prove that $AB^2 = AC^2 + BC^2$ by using vectors.




**42.** In the figure, $O$ is the centre of the circle, $M$ is the mid-point of chord $AB$. Prove that $OM \perp AB$.

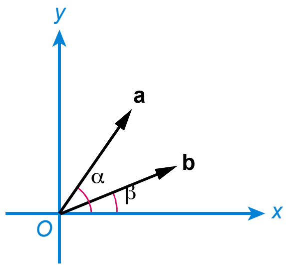


**43.** In the figure, $O$ is the centre of the circle, $AC$ is a diameter, $B$ is a point on the circumference. Consider $\overrightarrow{AB}$ and $\overrightarrow{BC}$, prove that $AB \perp BC$ by using vectors.

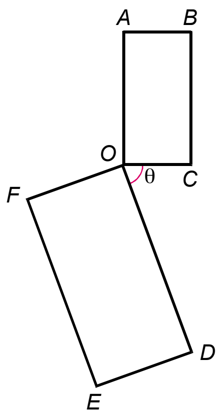



## *Level 3*

**44.** The figure shows unit vectors **a** and **b**. Prove that $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$ by using vectors.




**45.** In the figure, $OABC$ and $ODEF$ are squares. It is given that $OC = 1$, $OA = 2$, $OD = p$, $OF = q$ and $\angle COD = \theta$, where $0° < \theta < 90°$. Let $\overrightarrow{OC} = \mathbf{i}$ and $\overrightarrow{OA} = 2\mathbf{j}$.

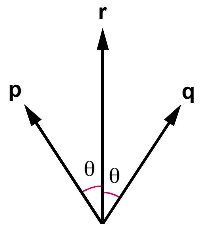



**(a)** **(i)** Express $\overrightarrow{OD}$ and $\overrightarrow{OF}$ in terms of $\theta$, $p$, $q$, $\mathbf{i}$ and $\mathbf{j}$.

**(ii)** Prove that $\overrightarrow{AD} = p\cos\theta\,\mathbf{i} - (p\sin\theta + 2)\,\mathbf{j}$ and $\overrightarrow{CF} = -(q\sin\theta + 1)\,\mathbf{i} - p\cos\theta\,\mathbf{j}$.

**(b)** If $\overrightarrow{AD}$ is perpendicular to $\overrightarrow{CF}$, prove that $p : q = 2 : 1$.

**46.** In the figure, $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ lie on the same plane. $\mathbf{p}$ and $\mathbf{q}$ are unit vectors, each of which makes an angle $\theta$ with $\mathbf{r}$, where $0 < \theta < \frac{\pi}{2}$.




**(a)** Prove that $\mathbf{p} \cdot \mathbf{r} = \mathbf{q} \cdot \mathbf{r}$.

**(b)** Prove that $-1 < \mathbf{p} \cdot \mathbf{q} < 1$.

**(c)** If $\mathbf{r} = x\mathbf{p} + y\mathbf{q}$, prove that $x = y$.

## EXERCISE 15B

### *Level 1*

**47.** Find the component and projection of $4\mathbf{i} - 3\mathbf{j}$ on $6\mathbf{i} + 8\mathbf{j}$.

**48.** Find the component and projection of $5\mathbf{i} + \mathbf{j}$ on $-2\mathbf{i} - 3\mathbf{j}$.

**49.** Find the component and projection of $-\mathbf{i} + \mathbf{j}$ on $2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$.

**50.** Find the component and projection of $\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$ on $\mathbf{i} + \mathbf{j} + 5\mathbf{k}$.

**51.** It is given that the coordinates of $A$ and $B$ are $(-1, 2, 3)$ and $(-2, 1, 3)$ respectively. Find the component and projection of $\overrightarrow{AB}$ on $3\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$.

**52.** It is given that the coordinates of $A$ and $B$ are $(-1, 1, 2)$ and $(3, -1, 1)$ respectively. Find the component and projection of $\overrightarrow{AB}$ on $\mathbf{i} - \mathbf{j} + \mathbf{k}$.

### *Level 2*

**53.** It is given that $O$ is the origin, and the coordinates of $A$ and $B$ are $(2, 6, -2)$ and $(6, 7, -6)$ respectively. Find the component and projection of $\overrightarrow{OA}$ on $\overrightarrow{OB}$.

**54.** It is given that $O$ is the origin, and the coordinates of $A$ and $B$ are $(2, -1, -2)$ and $(3, 1, -1)$ respectively. Find the component and projection of $\overrightarrow{OA}$ on $\overrightarrow{OB}$.

**55.** It is given that the coordinates of $A$, $B$ and $C$ are $(-1, 2, 3)$, $(1, 2, 0)$ and $(1, 6, -1)$ respectively. Find the component and projection of $\overrightarrow{AB}$ on $\overrightarrow{AC}$.

**56.** It is given that the coordinates of $A$, $B$ and $C$ are $(2, -1, 1)$, $(3, 0, 2)$ and $(3, -3, 4)$ respectively. Find the component and projection of $\overrightarrow{AB}$ on $\overrightarrow{AC}$.

**57.** It is given that $\mathbf{a} = \lambda\mathbf{i} - 3\mathbf{j} - 2\lambda\mathbf{k}$ and $\mathbf{b} = 6\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$. If the component of $\mathbf{a}$ on $\mathbf{b}$ is $\frac{13}{7}$, find the value of $\lambda$.

**58.** It is given that $\mathbf{a} = \mathbf{i} - 4\mathbf{j} + \mathbf{k}$ and $\mathbf{b} = -3\mathbf{i} + \mu\mathbf{j} + \mu\mathbf{k}$, where $\mu$ is an integer. If the component of $\mathbf{a}$ on $\mathbf{b}$ is $\frac{3\sqrt{17}}{17}$, find the value of $\mu$.

**59.** It is given that $O$ is the origin, and the coordinates of $A$, $B$ and $C$ are $(\lambda, \mu, 1)$, $(1, 2, -1)$ and $(4, 3, 4)$ respectively.

**(a)** **(i)** Express the component of $\overrightarrow{OA}$ on $\overrightarrow{OB}$ in terms of $\lambda$ and $\mu$.

**(ii)** Express the component of $\overrightarrow{OA}$ on $\overrightarrow{OC}$ in terms of $\lambda$ and $\mu$.

**(b)** If the component of $\overrightarrow{OA}$ on $\overrightarrow{OB}$ is $\frac{7}{\sqrt{6}}$ and that of $\overrightarrow{OA}$ on $\overrightarrow{OC}$ is $\frac{21}{\sqrt{41}}$,

**(i)** find the values of $\lambda$ and $\mu$.

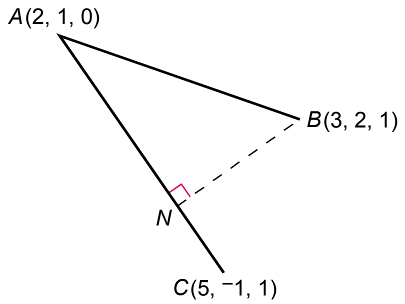
**(ii)** find projections of $\overrightarrow{OA}$ on $\overrightarrow{OB}$ and $\overrightarrow{OC}$ respectively.

**60.** It is given that $\mathbf{a}$ and $\mathbf{b}$ are non-zero vectors. If $\mathbf{p}$ is the projection of $\mathbf{a}$ on $\mathbf{b}$, prove that $(\mathbf{p} - \mathbf{a}) \cdot \mathbf{b} = 0$.

## *Level 3*

**61.** It is given that $\mathbf{a} = 3\mathbf{i} - \mathbf{j}$ and $\mathbf{b} = \mathbf{i} + 2\mathbf{j}$. If $\mathbf{c}$ is a two-dimensional unit vector such that the projections of $\mathbf{a}$ on $\mathbf{c}$ and $\mathbf{b}$ on $\mathbf{c}$ are the same, find $\mathbf{c}$.

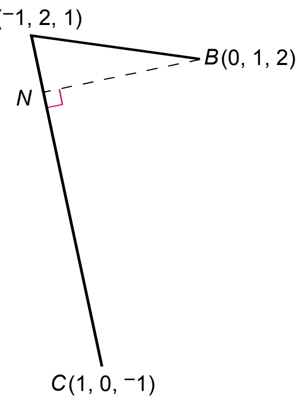
**62.** It is given that the coordinates of $A$, $B$ and $C$ are $(2, 1, 0)$, $(3, 2, 1)$ and $(5, -1, 1)$ respectively. $N$ is a point on $AC$ such that $BN \perp AC$.




**(a)** Find $\overrightarrow{AN}$.

**(b)** Find the coordinates of $N$.

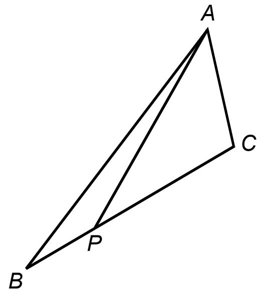
**63.** It is given that the coordinates of $A$, $B$ and $C$ are $(-1, 2, 1)$, $(0, 1, 2)$ and $(1, 0, -1)$ respectively. $N$ is a point on $AC$ such that $BN \perp AC$.




**(a)** Consider $\overrightarrow{AN}$, find the coordinates of $N$.

**(b)** Find the shortest distance from $B$ to $AC$.

**64.** In the figure, $ABC$ is a triangle. $P$ is a point on $BC$ such that $BP:PC = 1:2$. $AB = 5$, $BC = 4$ and $AC = 2$.




**(a)** Express $\overrightarrow{AP}$ in terms of $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

**(b)** Find the value of $\overrightarrow{BA} \cdot \overrightarrow{BC}$.

**(c)** Find the value of $\overrightarrow{AC} \cdot \overrightarrow{BC}$.

**(d)** Hence find the component of $\overrightarrow{AP}$ on $\overrightarrow{BC}$.

**65.** If $\mathbf{u} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$ and $\mathbf{v} = -\mathbf{i} - \mathbf{j} - 2\mathbf{k}$, find $\mathbf{u} \times \mathbf{v}$.

**66.** If $\mathbf{u} = -4\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ and $\mathbf{v} = 3\mathbf{i} - \mathbf{j} + 4\mathbf{k}$, find $\mathbf{u} \times \mathbf{v}$.

**67.** If $\mathbf{u} = 5\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}$ and $\mathbf{v} = \mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$, find $\mathbf{u} \times \mathbf{v}$ and $\mathbf{v} \times \mathbf{u}$.

**68.** If $\mathbf{u} = 7\mathbf{i} - \mathbf{j} - \mathbf{k}$ and $\mathbf{v} = -\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$, find $\mathbf{u} \times \mathbf{v}$ and $\mathbf{v} \times \mathbf{u}$.

**69.** Determine whether $\mathbf{a} = -\mathbf{i} + \mathbf{j} + 3\mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} - 2\mathbf{j} - 6\mathbf{k}$ are parallel.

**70.** Determine whether $\mathbf{a} = 2\mathbf{i} - 2\mathbf{j} - \mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ are parallel.

**71.** Find the area of the parallelogram formed by $\mathbf{a} = 2\mathbf{i} - \mathbf{j} - \mathbf{k}$ and $\mathbf{b} = \mathbf{i} + 3\mathbf{j} + \mathbf{k}$.

**72.** If three adjacent vertices of a parallelogram are $A(2, 1, -1)$, $B(-1, 2, 1)$ and $C(1, 1, 2)$ respectively, find the area of the parallelogram.

**73.** Find the area of the triangle formed by $\mathbf{a} = 4\mathbf{i} - \mathbf{k}$ and $\mathbf{b} = -\mathbf{i} - 2\mathbf{j} + \mathbf{k}$.

**74.** If the three vertices of a triangle are $A(1, 2, 3)$, $B(-2, 1, 4)$ and $C(2, 2, -1)$ respectively, find the area of the triangle.

(*) **75.** Determine whether $\mathbf{a} = \mathbf{i} - \mathbf{j} + 5\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$ and $\mathbf{c} = \mathbf{i} + \mathbf{j} - \mathbf{k}$ lie on the same plane.

(*) **76.** Determine whether $A(2, 1, 1)$, $B(3, 0, 3)$, $C(4, -1, 5)$ and $D(3, 3, 4)$ lie on the same plane.

(*) **77.** Find the volume of the parallelepiped formed by $\mathbf{a} = 4\mathbf{i} + 3\mathbf{j} + \mathbf{k}$, $\mathbf{b} = \mathbf{i} + 5\mathbf{j} - 2\mathbf{k}$ and $\mathbf{c} = \mathbf{i} - \mathbf{j} - \mathbf{k}$.

(*) **78.** It is given that the coordinates of $A$, $B$, $C$ and $D$ are $(-1, 2, 3)$, $(2, -1, 2)$, $(1, 1, 4)$ and $(2, -2, 5)$ respectively. Find the volume of the parallelepiped formed by $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AD}$.

(*) **79.** Find the volume of the tetrahedron formed by $\mathbf{a} = \mathbf{i} - 5\mathbf{j} + 6\mathbf{k}$, $\mathbf{b} = -2\mathbf{i} - \mathbf{j} - 3\mathbf{k}$ and $\mathbf{c} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$.

(*) **80.** It is given that $O$ is the origin, and the coordinates of $A$, $B$ and $C$ are $(2, 4, 3)$, $(1, -1, 3)$ and $(3, 4, 2)$ respectively.

**(a)** Find the volume of tetrahedron $OABC$.

**(b)** If $\Delta ABC$ is the base, find the height of the tetrahedron.

(*) **81.** It is given that the coordinates of $A$, $B$, $C$ and $D$ are $(2, 4, 5)$, $(1, 3, 2)$, $(-1, 2, 1)$ and $(1, -1, 2)$ respectively.

**(a)** Find the volume of tetrahedron $ABCD$.

**(b)** If $\Delta ABC$ is the base, find the height of the tetrahedron.

**82.** On a plane, the three vertices of triangle $ABC$ are $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ respectively. Prove that the area of the triangle is $\frac{1}{2}\left|x_1y_2 + x_2y_3 + x_3y_1 - x_1y_3 - x_2y_1 - x_3y_2\right|$.

**83.** It is given that $\mathbf{a} = \mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$.

**(a)** Find the unit vectors which are perpendicular to the plane formed by $\mathbf{a}$ and $\mathbf{b}$.

**(b)** Find the values of $\mathbf{a} \cdot \mathbf{b}$, $|\mathbf{a}|$ and $|\mathbf{b}|$.

**(c)** Hence verify that $|\mathbf{a} \times \mathbf{b}|^2 = |\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2$.

**84.** It is given that $\mathbf{a} = \mathbf{i} - \mathbf{j} + x\mathbf{k}$ and $\mathbf{b} = -2\mathbf{i} + x\mathbf{j} - 4\mathbf{k}$.

**(a)** Prove that $\mathbf{a} \times \mathbf{b} = (4 - x^2)\mathbf{i} - (2x - 4)\mathbf{j} + (x - 2)\mathbf{k}$.

**(b)** Hence, if $\mathbf{a} \,//\, \mathbf{b}$, find the value of $x$.

**85.** It is given that $\mathbf{a}$ and $\mathbf{b}$ are vectors. Prove that $(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{a}+\mathbf{b})=0$.

### *Level 2*

**86.** $A(2, 1, 6)$, $B(3, t, 4)$ and $C(1, 1, 6)$ are given. If the area of $\Delta ABC$ is $\frac{\sqrt{5}}{2}$, find the values of $t$.

**87.** $A(-1, 2, t)$, $B(2, 1, 3)$ and $C(1, 3, 2)$ are given. If the area of $\Delta ABC$ is $\frac{5\sqrt{2}}{2}$, find the values of $t$.

(*) **88.** If $\mathbf{a}=\lambda\mathbf{i}+\mathbf{j}+\mathbf{k}$, $\mathbf{b}=3\mathbf{i}+\lambda\mathbf{j}+3\mathbf{k}$ and $\mathbf{c}=-\mathbf{i}+\mathbf{k}$ lie on the same plane, find the values of $\lambda$.

(*) **89.** If $\mathbf{a}=4\mathbf{i}-4\mathbf{j}-3\mathbf{k}$, $\mathbf{b}=2\mathbf{i}-\lambda\mathbf{j}$ and $\mathbf{c}=\mathbf{i}-\mathbf{j}+\lambda\mathbf{k}$ lie on the same plane, find the values of $\lambda$.

**90.** It is given that the coordinates of $A$, $B$, $C$ and $D$ are $(1, 3, 3)$, $(-2, 1, 1)$, $(2, -2, -1)$ and $(2, 3, 4)$ respectively.

**(a)** Find the area of $\Delta ABC$.

**(b)** Find the component of $\overrightarrow{AD}$ on $\overrightarrow{AB}\times\overrightarrow{AC}$.

**(c)** Hence find the volume of tetrahedron $ABCD$.

**91.** It is given that the coordinates of $A$, $B$, $C$ and $D$ are $(2, -3, 4)$, $(1, 2, 1)$, $(3, -1, 1)$ and $(\lambda, -5, 4)$ respectively.

**(a)** Find the area of $\Delta ABC$.

**(b)** If the component of $\overrightarrow{AD}$ on $\overrightarrow{AB}\times\overrightarrow{AC}$ is $\frac{9\sqrt{474}}{79}$, find the value of $\lambda$.

**(c)** Hence find the volume of tetrahedron $ABCD$.

**92.** It is given that $|\mathbf{m}|=3$, $|\mathbf{n}|=2$ and the angle between $\mathbf{m}$ and $\mathbf{n}$ is $120°$. If $\mathbf{p}=2\mathbf{m}+\mathbf{n}$ and $\mathbf{q}=3\mathbf{m}-\mathbf{n}$, find

**(a)** the value of $\mathbf{m}\cdot\mathbf{n}$.

**(b)** the values of $|\mathbf{p}|$ and $|\mathbf{q}|$.

**(c)** the area of the parallelogram with $\mathbf{p}$ and $\mathbf{q}$ as its adjacent sides.

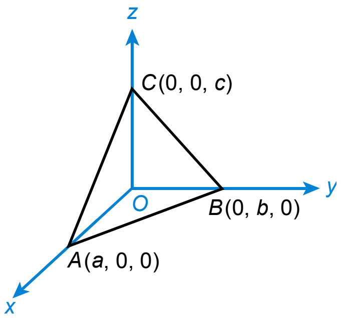
93. It is given that $|\mathbf{m}| = 4$, $|\mathbf{n}| = 3$ and the angle between $\mathbf{m}$ and $\mathbf{n}$ is $\frac{\pi}{3}$. If $\mathbf{p} = 4\mathbf{m} - \mathbf{n}$ and $\mathbf{q} = \mathbf{m} + 2\mathbf{n}$, find

**(a)** the value of $\mathbf{m} \cdot \mathbf{n}$.

**(b)** the values of $|\mathbf{p}|$ and $|\mathbf{q}|$.

**(c)** the area of the parallelogram with $\mathbf{p}$ and $\mathbf{q}$ as its adjacent sides.

(*) 94. In the figure, find the perpendicular distance from the origin $O$ to plane $ABC$.




95. Let $\mathbf{m}$ and $\mathbf{n}$ be three-dimensional vectors and $\lambda$ is a real number. It is given that $\begin{cases} \mathbf{u} = \lambda\mathbf{n} + (1-\lambda)\mathbf{m} \\ \mathbf{v} = 3(1-\lambda)\mathbf{n} - \lambda\mathbf{m} \end{cases}$.

**(a)** Prove that $\mathbf{u} \times \mathbf{v} = (4\lambda^2 - 6\lambda + 3)\mathbf{m} \times \mathbf{n}$.

**(b)** Suppose $|\mathbf{m}| = 5$, $|\mathbf{n}| = 3$ and the angle between $\mathbf{m}$ and $\mathbf{n}$ is $30°$.

**(i)** Evaluate $|\mathbf{m} \times \mathbf{n}|$.

**(ii)** Find the smallest area of the parallelogram with adjacent sides $\mathbf{u}$ and $\mathbf{v}$ as $\lambda$ varies.

96. Let $\mathbf{m}$ and $\mathbf{n}$ be three-dimensional vectors and $\lambda$ is a real number. It is given that $\begin{cases} \mathbf{u} = \lambda\mathbf{n} + (2-\lambda)\mathbf{m} \\ \mathbf{v} = 3(2-\lambda)\mathbf{n} - \lambda\mathbf{m} \end{cases}$.

**(a)** Prove that $\mathbf{u} \times \mathbf{v} = (4\lambda^2 - 12\lambda + 12)\mathbf{m} \times \mathbf{n}$.

**(b)** Suppose $|\mathbf{m}| = 2\sqrt{2}$, $|\mathbf{n}| = 3$ and the angle between $\mathbf{m}$ and $\mathbf{n}$ is $\frac{\pi}{4}$.

**(i)** Evaluate $|\mathbf{m} \times \mathbf{n}|$.

**(ii)** Find the smallest area of the parallelogram with adjacent sides $\mathbf{u}$ and $\mathbf{v}$, and the corresponding value of $\lambda$ as $\lambda$ varies.

**97.** Prove that $[(\mathbf{b}+\mathbf{c})\times(\mathbf{c}+\mathbf{a})]\cdot(\mathbf{a}+\mathbf{b}) = 2(\mathbf{c}\times\mathbf{a})\cdot\mathbf{b}$.

**98.** It is given that **a**, **b** and **c** are vectors and $\mathbf{a}+\mathbf{b}+\mathbf{c}=\mathbf{0}$.

**(a)** Prove that $\mathbf{a}\times\mathbf{b} = \mathbf{b}\times\mathbf{c} = \mathbf{c}\times\mathbf{a}$.

**(b)** If **a**, **b** and **c** are parallel to each other, find $\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}$.

### *Level 3*

**99.** In $\Delta ABC$, let $\overrightarrow{AB}=\mathbf{c}$, $\overrightarrow{BC}=\mathbf{a}$ and $\overrightarrow{CA}=\mathbf{b}$.

**(a)** Prove that $\mathbf{a}\times\mathbf{b} = \mathbf{b}\times\mathbf{c} = \mathbf{c}\times\mathbf{a}$.

**(b)** Hence prove that $\dfrac{|\mathbf{a}|}{\sin A} = \dfrac{|\mathbf{b}|}{\sin B} = \dfrac{|\mathbf{c}|}{\sin C}$.

**(*)** **100.** It is given that **a**, **b** and **c** are non-zero vectors, $x$, $y$ and $z$ are non-zero constants. If $x\mathbf{a}\times\mathbf{b}+y\mathbf{b}\times\mathbf{c}+z\mathbf{c}\times\mathbf{a}=\mathbf{0}$, prove that **a**, **b** and **c** lie on the same plane.

**101.** Let **a**, **b** and **c** be the position vectors of $A$, $B$ and $C$ respectively. If $\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}=\mathbf{0}$, prove that $A$, $B$ and $C$ are collinear.

**102.** It is given that **a**, **b** and **c** are vectors.

**(a)** Prove that $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c}$.

**(b)** Hence prove that $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})+\mathbf{b}\times(\mathbf{c}\times\mathbf{a})+\mathbf{c}\times(\mathbf{a}\times\mathbf{b})=\mathbf{0}$.