Chapter 13 Matrices and Systems of Linear Equations

EXERCISE 13A

Level 1

1. It is given that $A = \begin{pmatrix} 3 & 7 & 8 \\ 2 & x & 5 \end{pmatrix}$ and $B = \begin{pmatrix} z & 7 & y \\ 2 & 2x-1 & 5 \end{pmatrix}$. If $A = B$, find the values of x, y and z.

2. Evaluate each of the following expressions.

   (a) $2\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$

   (b) $3\begin{pmatrix} 1 & 3 & 2 \\ 2 & 5 & -2 \end{pmatrix} + 2\begin{pmatrix} 0 & 2 & -1 \\ 1 & 0 & 2 \end{pmatrix}$

3. Evaluate each of the following expressions.

   (a) $\begin{pmatrix} 7 & -2 \\ -1 & 3 \end{pmatrix} + 2\begin{pmatrix} 2 & 1 \\ -2 & -3 \end{pmatrix}$

   (b) $5\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} - 2\begin{pmatrix} 3 \\ 2 \\ -5 \end{pmatrix}$

4. Let $A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}$ and $C = \begin{pmatrix} 2 & -2 \\ -1 & 4 \end{pmatrix}$.

   (a) Find $A + B$.

   (b) Find $2A - B + C$.

5. Let $P = \begin{pmatrix} 2 & 0 \\ 1 & -1 \\ 0 & 1 \end{pmatrix}$, $Q = \begin{pmatrix} 0 & 1 \\ 3 & 2 \\ 2 & -1 \end{pmatrix}$ and $R = \begin{pmatrix} 1 & 0 \\ 2 & 0 \\ -2 & 5 \end{pmatrix}$.

   (a) Find $3P + Q - 2R$.

   (b) Find $2P - Q + 2R - R + P + 3Q - 4R$.

(*) Out syl after HKDSE 2022

Properties of determinants:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = -\begin{vmatrix} c_1 & b_1 & a_1 \\ c_2 & b_2 & a_2 \\ c_3 & b_3 & a_3 \end{vmatrix}$$

$$\begin{vmatrix} a_1 & b_1 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & 0 \end{vmatrix} = 0$$

$$\begin{vmatrix} a_1 & kb_1 & c_1 \\ a_2 & kb_2 & c_2 \\ a_3 & kb_3 & c_3 \end{vmatrix} = k\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$\begin{vmatrix} a_1 & b_1 & kb_1 \\ a_2 & b_2 & kb_2 \\ a_3 & b_3 & kb_3 \end{vmatrix} = 0$$

$$\begin{vmatrix} a_1 + a_1' & b_1 & c_1 \\ a_2 + a_2' & b_2 & c_2 \\ a_3 + a_3' & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1' & b_1 & c_1 \\ a_2' & b_2 & c_2 \\ a_3' & b_3 & c_3 \end{vmatrix}$$

$$\begin{vmatrix} a_1 + kb_1 & b_1 & c_1 \\ a_2 + kb_2 & b_2 & c_2 \\ a_3 + kb_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

6. Evaluate $\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix} - \frac{1}{3}\begin{pmatrix} 6 & 0 & 9 \\ -3 & 3 & 0 \\ 27 & 18 & 15 \end{pmatrix} + 4\begin{pmatrix} 2 & 0 & 0 \\ \frac{1}{2} & 0 & -\frac{1}{4} \\ -1 & 1 & \frac{1}{2} \end{pmatrix}$.

7. Let $A = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $B = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$. Find the matrix C such that $A - C = B$.

8. Let $P = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$ and $Q = \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}$. Find the matrix R such that $P - R = 2Q$.

Level 2

9. It is given that $A = \begin{pmatrix} x^2+1 & 4 \\ -1 & y \end{pmatrix}$ and $B = \begin{pmatrix} 2x & 4 \\ -1 & 2x \end{pmatrix}$. If $A = B$, find the values of x and y.

10. Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 1 & -1 \end{pmatrix}$. Find the values of a, b, c and d.

11. Let $\begin{pmatrix} x & 1 \\ 1 & y \end{pmatrix} + \begin{pmatrix} 1 & 4 \\ 2 & 2y \end{pmatrix} = \begin{pmatrix} y & 5 \\ 3 & 9 \end{pmatrix}$. Find the values of x and y.

12. Let $\begin{pmatrix} x & -1 \\ 2x & 3x \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 2y & -1 \\ 4y & 4y \end{pmatrix}$. Find the values of x and y.

13. Evaluate $\begin{pmatrix} \sin^2\theta & -\tan^2\theta \\ \cos\theta & 0 \end{pmatrix} + \begin{pmatrix} \cos^2\theta & \sec^2\theta \\ \sin(\frac{3\pi}{2} - \theta) & 0 \end{pmatrix}$.

14. For $m \times n$ matrices A, B and scalar λ, show that $\lambda(A \pm B) = \lambda A \pm \lambda B$.

15. For $m \times n$ matrix A and scalars λ, μ, show that $(\lambda \pm \mu)A = \lambda A \pm \mu A$.

Level 3

16. Let A and B be 2×2 matrices. If $A + B = \begin{pmatrix} 3 & 3 \\ 4 & 5 \end{pmatrix}$ and $A - B = \begin{pmatrix} 1 & -1 \\ -2 & 1 \end{pmatrix}$, find A and B.

17. Let P and Q be 2×3 matrices. If $P + 2Q = \begin{pmatrix} 1 & 2 & 4 \\ 0 & 5 & 1 \end{pmatrix}$ and $P - Q = \begin{pmatrix} 1 & -1 & 1 \\ 0 & -1 & 1 \end{pmatrix}$, find P and Q.

EXERCISE 13B

Level 1

18. Let $A = \begin{pmatrix} 2 & 6 \end{pmatrix}$ and $B = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$. Find the following products of matrices.

(a) AB

(b) BA

19. Let $A = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$. Find the following products of matrices.

(a) AB

(b) BA

20. Evaluate $\begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & 3 \end{pmatrix}\begin{pmatrix} 0 & 3 \\ 1 & 3 \\ 0 & 4 \end{pmatrix}$.

21. Evaluate $\begin{pmatrix} 1 & 1 & 3 \\ -1 & 0 & 2 \\ 0 & 3 & 1 \end{pmatrix}\begin{pmatrix} 0 & 2 & -1 \\ 1 & 1 & 4 \\ 3 & 0 & -1 \end{pmatrix}$.

22. Evaluate $\begin{pmatrix} 2 & 4 & -1 \\ 1 & 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

23. Let $\begin{pmatrix} 2 & x & 0 \\ 1 & 0 & 2 \end{pmatrix}\begin{pmatrix} 4 & 1 \\ 1 & 0 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 11 & 2 \\ 8 & y \end{pmatrix}$. Find the values of x and y.

24. Let $P = \begin{pmatrix} x & -y \\ y & x \end{pmatrix}$ and $Q = \begin{pmatrix} m & -n \\ n & m \end{pmatrix}$. Prove that $PQ = QP$.

Level 2

25. Let $A = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$.

(a) Find $A + 2B$.

(b) If $(A + 2B)\begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ y \end{pmatrix}$, find the values of x and y.

26. Let $\begin{pmatrix} 3 & 6 \\ x & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 2x & 2 \end{pmatrix} = \begin{pmatrix} 2y & y \\ 4x & x+2 \end{pmatrix}$. Find the values of x and y.

27. Let $\begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & -1 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 0 & 5 \\ 1 & x \end{pmatrix} = \begin{pmatrix} 4 & 8-x \\ y-2z & y \end{pmatrix}$. Find the values of x, y and z.

28. Let $A = \begin{pmatrix} 0 & x \\ y & z \end{pmatrix}$ and $A\begin{pmatrix} 8 \\ 4 \end{pmatrix} = \begin{pmatrix} 8 \\ 52 \end{pmatrix}$.

(a) Find the value of x.

(b) Prove that $2y + z = 13$. Also, if $y + z = 8$, find A.

29. **(a)** Prove that $\begin{pmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{pmatrix}\begin{pmatrix} \cos y & -\sin y \\ \sin y & \cos y \end{pmatrix} = \begin{pmatrix} \cos(x+y) & -\sin(x+y) \\ \sin(x+y) & \cos(x+y) \end{pmatrix}$.

(b) Hence find $\begin{pmatrix} \cos 45° & -\sin 45° \\ \sin 45° & \cos 45° \end{pmatrix}\begin{pmatrix} \cos 45° & -\sin 45° \\ \sin 45° & \cos 45° \end{pmatrix}$.

Level 3

30. Let $A = \begin{pmatrix} x & 1 \\ -1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} y & 4 \\ 1 & x \end{pmatrix}$ and $C = \begin{pmatrix} 7 & 15 \\ -y & z \end{pmatrix}$. If $AB = C$, find C.

31. Let $A = \begin{pmatrix} 2 & -1 & 3 \\ x & 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ y & 0 \\ 1 & 3 \end{pmatrix}$, $C = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$ and $D = \begin{pmatrix} 21 & 24 \\ 4y & z \end{pmatrix}$. If $ABC = D$, find D.

EXERCISE 13C

Level 1

32. Find each of the following products of matrices.

(a) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 7 \\ -1 & 3 \end{pmatrix}$

(b) $\begin{pmatrix} 2 & 0 & 1 \\ 1 & 4 & 2 \\ 3 & -1 & -2 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

33. Find each of the following products of matrices.

(a) $\begin{pmatrix} 7 & 8 \\ 9 & 10 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 7 & 2 & 4 \\ -1 & 8 & -9 \\ 11 & 5 & 2 \end{pmatrix}$

34. In each of the following, find the product of matrices. Hence show that the two matrices are the inverses of each other.

(a) $\begin{pmatrix} 13 & 3 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} 1 & -3 \\ -4 & 13 \end{pmatrix}$

(b) $\begin{pmatrix} 2 & 1 & -1 \\ -3 & 2 & 1 \\ 0 & 5 & 0 \end{pmatrix}\begin{pmatrix} -1 & -1 & \frac{3}{5} \\ 0 & 0 & \frac{1}{5} \\ -3 & -2 & \frac{7}{5} \end{pmatrix}$

35. **(a)** Find $\begin{pmatrix} 7 & 9 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 4 & -9 \\ -3 & 7 \end{pmatrix}$.

(b) Hence find $\begin{pmatrix} 7 & 9 \\ 3 & 4 \end{pmatrix}^{-1}$ and $\begin{pmatrix} 4 & -9 \\ -3 & 7 \end{pmatrix}^{-1}$.

36. **(a)** Find $\begin{pmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & -2 \end{pmatrix}$.

(b) Hence find $\begin{pmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix}^{-1}$ and $\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & -2 \end{pmatrix}^{-1}$.

37. Find $\begin{pmatrix} 1 & 0 \\ -3 & 2 \end{pmatrix}^{3}$.

38. Find $\begin{pmatrix} 4 & -1 \\ 3 & 0 \end{pmatrix}^{4}$.

Level 2

39. **(a)** Prove that $\begin{pmatrix} k & 1 \\ 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -k \end{pmatrix}$.

(b) Hence find $\begin{pmatrix} 8 & 1 \\ 1 & 0 \end{pmatrix}^{-1}$.

40. Let $A = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}$.

(a) Find A^2.

(b) Hence find A^{-1}.

(c) Find$(A^{-1})^2$.

41. Let $A = \begin{pmatrix} x & 1 \\ 1 & y \end{pmatrix}$. If $A^2 = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$, find the values of x and y.

42. Let $A = \begin{pmatrix} 4 & x \\ 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -3 \\ -1 & y \end{pmatrix}$.

(a) Find AB.

(b) If B is the inverse of A, find the values of x and y.

43. Let $A = \begin{pmatrix} 3 & x \\ 4 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & y \\ 4 & -3 \end{pmatrix}$.

(a) Find AB.

(b) If B is the inverse of A, find the values of x and y.

44. Let $A = \begin{pmatrix} 5 & 2 \\ -3 & 1 \end{pmatrix}$. Find the values of p and q such that $A^2 + pA + qI = \mathbf{0}$.

45. Let $A = \begin{pmatrix} m & 0 \\ 0 & -m \end{pmatrix}$, where $m \neq 0$.

(a) Prove that $A^2 - m^2 I = \mathbf{0}$.

(b) Hence prove that $A^3 + A^2 - 4A - 4I = (m^2 - 4)A + (m^2 - 4)I$.

(c) If $A^3 + A^2 - 4A - 4I = \mathbf{0}$, find A.

46. Let $A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$.

(a) Prove that $A^2 - 4A + 3I = \mathbf{0}$.

(b) Find A^{-1}.

47. Let $A = \begin{pmatrix} 2 & 3 \\ 5 & 3 \end{pmatrix}$. Find A^{-1}.

48. Prove, by mathematical induction, that for all positive integers n, $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}^n = \begin{pmatrix} a^n & 0 \\ 0 & b^n \end{pmatrix}$.

Level 3

49. Prove, by mathematical induction, that for all positive integers n, $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}^n = \begin{pmatrix} 3^{n-1} & 3^{n-1} & 3^{n-1} \\ 3^{n-1} & 3^{n-1} & 3^{n-1} \\ 3^{n-1} & 3^{n-1} & 3^{n-1} \end{pmatrix}$.

50. **(a)** Prove, by mathematical induction, that for all positive integers n,

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}^n = \begin{pmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{pmatrix}.$$

(b) Find $\begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}^{2010}$.

51. Let A be a square matrix such that $A^2 - 2A - I = \mathbf{0}$. Prove that A is a non-singular matrix.

Exercise 13D

Level 1

52. Let $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 3 \end{pmatrix}$. Find AA^T and $A^T A$.

53. Let $A = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 2 & 1 \end{pmatrix}$. Find AA^T and $A^T A$.

54. Let $A = \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix}$. Find AA^T and $A^T A$.

55. Let $A = \begin{pmatrix} 2 & -1 & 3 \end{pmatrix}$. Find AA^T and $A^T A$.

56. It is given that A, B, C and D are square matrices, where $AB^T C = D^2$, $|A| = 3$, $|C| = 6$ and $|D| = 12$. Find the value of $|B|$.

57. It is given that A, B, C and D are square matrices, where $AB^{-1} = C^3 D^T$, $|A| = 48$, $|B| = 2$ and $|D| = 3$. Find the value of $|C|$.

58. It is given that P, Q and R are square matrices of order 2, where $P^2 Q^T = R$, $PQ = 2I$ and $|Q| = 4$. Find the values of $|P|$ and $|R|$.

59. It is given that P, Q and R are square matrices of order 2, where R is invertible, $P^2 R = PQ^T$, $P^{-1} Q = \sqrt{3} I$ and $|Q| = 15$. Find the values of $|P|$ and $|R^{-1}|$.

60. Find the value of the determinant of each of the following matrices.

(a) $A = \begin{pmatrix} 1 & 2 \\ 4 & 8 \end{pmatrix}$ **(b)** $B = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 0 & 1 \\ 3 & -1 & 0 \end{pmatrix}$

61. Find the adjoint matrix of each of the following matrices.

(a) $A = \begin{pmatrix} 2 & 3 \\ -1 & 4 \end{pmatrix}$ **(b)** $A = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 2 & 3 \\ 0 & 1 & 4 \end{pmatrix}$

62. Find the adjoint matrix of each of the following matrices.

(a) $A = \begin{pmatrix} 5 & -2 \\ 1 & -3 \end{pmatrix}$ **(b)** $A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix}$

63. Find the inverse of $B = \begin{pmatrix} 7 & 4 \\ 5 & 3 \end{pmatrix}$.

64. Find the inverse of $B = \begin{pmatrix} 9 & 1 \\ 8 & 1 \end{pmatrix}$.

65. Find the inverse of $B = \begin{pmatrix} 2 & -3 \\ 4 & 7 \end{pmatrix}$.

66. Find the inverse of $B = \begin{pmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{pmatrix}$.

67. Find the inverse of $B = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 2 & 3 & 2 \end{pmatrix}$.

68. Find the inverse of $B = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 2 & 1 \\ 5 & 5 & 3 \end{pmatrix}$.

69. Find the inverse of $B = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 2 & 3 \\ 4 & 0 & 2 \end{pmatrix}$.

70. Find the inverse of $B = \begin{pmatrix} 3 & -1 & 4 \\ 2 & 1 & 3 \\ 4 & 2 & 1 \end{pmatrix}$.

71. Find the inverse of $B = \begin{pmatrix} -2 & 5 & 2 \\ 4 & 2 & 1 \\ -3 & 1 & 1 \end{pmatrix}$.

Level 2

72. Let $P = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \end{pmatrix}$ and $Q = PP^T$. Find Q^{-1}.

73. Let $A = \begin{pmatrix} 0 & 2 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & 4 \end{pmatrix}$ and $B = A^T A$. Find B^{-1}.

74. Let $A = \begin{pmatrix} 2 & -4 \\ -2 & 5 \end{pmatrix}$. Find $A^{-1} + 2(A^2)^{-1}$.

75. Let $A = \begin{pmatrix} 2 & x \\ 2 & y \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}$.

(a) If $\det A = \det B$, express y in terms of x.

(b) If $AP = P$, find A for each of the following conditions.

(i) $P = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ **(ii)** $P = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

76. If the value of the determinant of $\begin{pmatrix} 3 & 2 & -1 \\ -1 & 1 & 2x \\ x & 0 & 1 \end{pmatrix}$ is 8, find the values of x.

77. It is given that $A = \begin{pmatrix} 6 & a \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & b \\ 3 & 6 \end{pmatrix}$, where A and B are singular matrices.

(a) Find the values of a and b.

(b) Find $(A - B)^{-1}$.

78. It is given that $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $A\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$ and $A\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 24 \\ 5 \end{pmatrix}$.

(a) Find the values of a, b, c and d.

(b) Find A^{-1}.

79. Let $A = \begin{pmatrix} 6 & 4 \\ a & 4 + \frac{a}{2} \end{pmatrix}$.

(a) If A is a singular matrix, find the value of a.

(b) Is A^2 invertible?

80. Let $A = \begin{pmatrix} 5 & -3 \\ a & -a-2 \end{pmatrix}$.

(a) If A is a singular matrix, find the value of a.

(b) Is A^2 invertible?

81. If $AA^T = \begin{pmatrix} 9 & 6 \\ 6 & 4 \end{pmatrix}$ and the dimensions of A are 2×1, find A.

82. If $AA^T = \begin{pmatrix} 16 & -4 \\ -4 & 1 \end{pmatrix}$ and the dimensions of A are 2×1, find A.

83. Let A and B be two non-singular square matrices of order n. If $A^T = A^{-1}$, prove that $(ABA^T)^{-1} = AB^{-1}A^T$.

Level 3

84. Let A and B be two non-singular square matrices of order n.

(a) Prove that $A^{-1}BA$ is a non-singular matrix.

(b) Simplify $(A^{-1}BA)^{-1}$.

85. Let A and B be two non-singular square matrices of order n.

(a) Prove that $A^T BA$ is a non-singular matrix.

(b) Simplify $(A^T BA)^T$.

(*) **86.** If x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9 form an arithmetic sequence, prove that $\begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{pmatrix}$ is not invertible.

87. Let $A = \begin{pmatrix} 2 & 2 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & -2 \\ 0 & 1 \end{pmatrix}$.

(a) Find $A^{-1}BA$.

(b) It is given that for all positive integers n, $\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}^n = \begin{pmatrix} p^n & 0 \\ 0 & q^n \end{pmatrix}$, where p and q are real numbers. Find $B^{2\,010}$.

88. Let $A = \begin{pmatrix} k & 1 \\ 1 & k \end{pmatrix}$ and $B = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

(a) Find B^{-1}.

(b) Prove that $B^{-1}AB = \begin{pmatrix} k + \sin 2\theta & \cos 2\theta \\ \cos 2\theta & k - \sin 2\theta \end{pmatrix}$.

(c) If $k = 2$, find the values of θ, x and y such that $B^{-1}AB = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$, where $0 < \theta < \frac{\pi}{2}$.

(d) It is given that for all positive integers n, $\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}^n = \begin{pmatrix} p^n & 0 \\ 0 & q^n \end{pmatrix}$, where p and q are real number. According to the values obtained in **(c)**, find A^9.

EXERCISE 13E

Level 1

89. Solve the system $\begin{cases} x - y = 2 \\ 6x - 5y = 3 \end{cases}$ by inverse matrix.

90. Solve the system $\begin{cases} 3x - 2y = 1 \\ x - 3y = 5 \end{cases}$ by inverse matrix.

91. Solve the system $\begin{cases} 2x + y = 1 \\ 3x - 4y = 7 \end{cases}$ by inverse matrix.

92. Solve the system $\begin{cases} 3x + y = 4 \\ 4x - 2y = -3 \end{cases}$ by inverse matrix.

93. Solve the system $\begin{cases} x + 2y - z = 5 \\ x - y + 2z = 2 \\ x - 3y = -4 \end{cases}$ by inverse matrix.

94. Solve the system $\begin{cases} x+2y+2z=5 \\ x-y-3z=-2 \\ x-2y-2z=1 \end{cases}$ by inverse matrix.

95. Solve the system $\begin{cases} 2x-2y-z=-1 \\ 3x-2y+3z=-1 \\ x-y+2z=-3 \end{cases}$ by inverse matrix.

96. Solve the system $\begin{cases} 4x-3y-2z=-1 \\ 2x-6y+3z=2 \\ 2x-9y+5z=3 \end{cases}$ by inverse matrix.

97. Solve the system $\begin{cases} x-2y+z=2 \\ -x+y-z=1 \\ x+3z=5 \end{cases}$ by Gaussian elimination.

98. Solve the system $\begin{cases} 3x-2y+5z=-2 \\ 2x+3y-4z=-3 \\ x+y-z=-1 \end{cases}$ by Gaussian elimination.

99. Solve the system $\begin{cases} 4x-y+3z=12 \\ x+2y+4z=5 \\ x-3y-5z=-3 \end{cases}$ by Gaussian elimination.

100. Solve the system $\begin{cases} 2x+2y-5z=-6 \\ 2x+5y-2z=9 \\ x+4y+z=13 \end{cases}$ by Gauss-Jordan elimination.

101. Solve the system $\begin{cases} x+y=3 \\ 2x-y+z=6 \\ 3x-2y+2z=10 \end{cases}$ by Gauss-Jordan elimination.

102. Solve the system $\begin{cases} 5x+4y+2z=-2 \\ 4x+2y+3z=8 \\ 2x-5y-z=-5 \end{cases}$ by Gauss-Jordan elimination.

103. Solve the system $\begin{cases} x - y + z = 2 \\ 2x - y + 3z = 5 \\ 3x - 2y + 4z = 9 \end{cases}$.

104. Solve the system $\begin{cases} x + z = 1 \\ y + z = 1 \\ x + y + 2z = 2 \end{cases}$.

105. Solve the system $\begin{cases} x + 2y - z = 1 \\ 2x + 5y - z = 3 \\ x + 3y = 2 \end{cases}$.

Level 2

106. If the system $\begin{cases} x - 5y + 4z = 1 \\ 2x - y - z = -1 \\ x - 3y + 2z = h \end{cases}$ has infinitely many solutions, find the value of h.

107. If the system $\begin{cases} x - 2y + 3z = 2 \\ 2x - y + z = h \\ x + y + kz = 3 \end{cases}$ has infinitely many solutions, find the values of h and k.

108. Find the values of k such that the following homogeneous linear system has non-trivial solutions.

$$\begin{cases} 6x + y - kz = 0 \\ 4x + y - 3z = 0 \\ (k + 4)x + 5z = 0 \end{cases}$$

109. Find the values of k such that the following homogeneous linear system has non-trivial solutions.

$$\begin{cases} -x + (k + 2)y - 6z = 0 \\ kx + 3y + 4z = 0 \\ x + 7y - 2z = 0 \end{cases}$$

(*) **110.** **(a)** Prove that $\begin{vmatrix} a^2 & b^2 & c^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} = (a-b)(b-c)(a-c)$.

(b) Consider the following linear systems (where a, b, c and k are real numbers):

$$(*)\begin{cases} a^2x + b^2y + c^2z = 0 \\ ax + by + cz = 0 \\ x + y + z = 0 \end{cases} \quad \text{and} \quad (**)\begin{cases} a^2x + b^2y + c^2z = k^2 \\ ax + by + cz = k \\ x + y + z = 1 \end{cases}$$

(i) Find the conditions of a, b and c such that (*) has non-trivial solutions.

(ii) Find the conditions of a, b and c such that (**) has a unique solution.

111. Consider the linear system $(*)\begin{cases} x - 2y + z = 2 \\ 2x - y + 5z = 13 \\ x - y + pz = q \end{cases}$, where p and q are real numbers.

(a) If (*) has infinitely many solutions, find the values of p and q.

(b) According to the values of p and q obtained in **(a)**, find the solutions of (*).

112. Consider the homogeneous linear system $(*)\begin{cases} 2x + (a-1)y - z = 0 \\ ax + 3y - 4z = 0 \\ -2x + 2y + z = 0 \end{cases}$, where a is a real number.

(a) Find the values of a such that (*) has non-trivial solutions.

(b) According to the values of a obtained in **(a)**, find the solutions of (*).

113. Consider the homogeneous linear system $(*)\begin{cases} kx - y + z = 0 \\ 7x + 5y + kz = 0 \\ x + y = 0 \end{cases}$, where k is a real number.

(a) Find the values of k such that (*) has non-trivial solutions.

(b) According to the values of k obtained in **(a)**, find the solutions of (*).

Level 3

114. Consider the linear system (*) $\begin{cases} 2x + y + 3z = kx \\ x - 2y + z = ky \\ 3x - y + 4z = kz \end{cases}$, where k is an integer.

(a) Rewrite (*) as a homogeneous linear system (**).

(b) Find the value of k such that (**) has non-trivial solutions. Then solve (**) for the value of k obtained.

115. Consider the linear system (*) $\begin{cases} x + az = b \\ 2x - y - z = 0 \\ -x + ay + 2z = 1 \end{cases}$, where a and b are real numbers.

(a) Find the range of values of a such that (*) has a unique solution. Solve (*) when it has a unique solution.

(b) Let $a = -1$. Find the value of b such that (*) has solutions. Solve (*) for the value of b obtained.

(c) Suppose that (x, y, z) satisfies $\begin{cases} x + z = 1 \\ 2x - y - z = 0 \\ -x + y + 2z = 1 \end{cases}$. Find the least value of $x^2 - y + z$ and the corresponding values of x, y, z.

(*) **116.** Consider the following linear systems (where a, b and c are distinct real numbers and $ab + bc + ac \neq 0$):

(*) $\begin{cases} x + a^2y + a^3z = 0 \\ x + b^2y + b^3z = 0 \\ x + c^2y + c^3z = 0 \end{cases}$ and (**) $\begin{cases} x + a^2y + a^3z = 1 + a^2 + a^3 \\ x + b^2y + b^3z = 1 + b^2 + b^3 \\ x + c^2y + c^3z = 1 + c^2 + c^3 \end{cases}$

(a) Prove that (*) has no non-trivial solutions.

(b) Find the solution of (**).