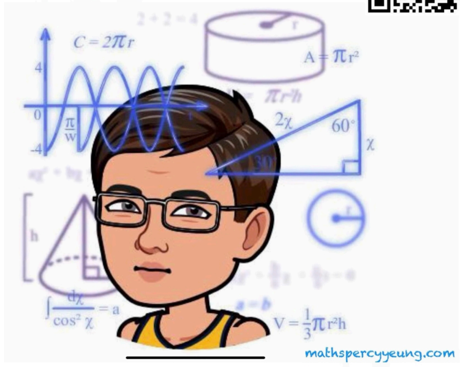
2022-2023 S6
MOCK EXAM
MATH EP
M2

2022 – 2023
S6 Mock Examination



MATHEMATICS Extended Part

Module 2 (Algebra and Calculus)

Question–Answer Book

5th January, 2023
8:15 am – 10:45 am (2 hours 30 minutes)
This paper must be answered in English

INSTRUCTIONS

1. Write your name, class and class number in the spaces provided on this cover.

2. This paper consists of TWO sections, A and B.

3. Attempt ALL questions in this paper. Write your answers in the spaces provided in this Question – Answer Book. Do not write in the margins. Answers written in the margins will not be marked.

4. Unless otherwise specified, all working must be clearly shown.

5. Unless otherwise specified, numerical answers must be exact.

6. The diagrams in this paper are not necessarily drawn to scale.

Sections	Marks
A Total	**/50**
B Total	**/50**
TOTAL	**/100**

FORMULAS FOR REFERENCE

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$	$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$
$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$	$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$
$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$	$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$
$2\sin A \cos B = \sin(A+B) + \sin(A-B)$	$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$
$2\cos A \cos B = \cos(A+B) + \cos(A-B)$	
$2\sin A \sin B = \cos(A-B) - \cos(A+B)$	

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Section A (50 marks)

1. Find $\frac{d}{dx}(\ln 2x)$ from first principles. (4 marks)

2. The coefficient of x^2 in the expansion of $(1-4x)^n$ is 240 , where n is a positive integer. Find

(a) n ,

(b) the coefficient of x^4 in the expansion of $(1-4x)^n\left(1+\dfrac{2}{x}\right)^5$.

(6 marks)

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3. (a) Using mathematical induction, prove that $\sum_{k=1}^{n} \frac{(-1)^{k+1}(2k+1)}{k(k+1)} = \frac{n+1+(-1)^{n+1}}{n+1}$ for all positive integers n.

(b) Using (a), evaluate $\sum_{k=3}^{321} \frac{(-1)^{k}(2k+1)}{25k(k+1)}$.

(7 marks)

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4. Find the equation of tangent to the curve $xe^y + y^2 \sin x = 6$ at the point where the curve cuts the x-axis. (5 marks)

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5. (a) Prove that $\cot\theta + \csc\theta = \cot\frac{\theta}{2}$ and evaluate $\cot\frac{\pi}{12}$.

(b) Find the value of $\csc\frac{4\pi}{15} + \csc\frac{8\pi}{15} + \csc\frac{16\pi}{15} + \csc\frac{32\pi}{15}$.

(6 marks)

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6. Consider the system of linear equations in x, y, z

$$(E): \begin{cases} hx + 3y + z = a \\ h^2x + 9y + z = b \\ h^3x + 27y + z = c \end{cases}$$, where a, b, c and h are real numbers.

(a) Assume that (E) has a unique solution.

(i) Find the range of values of h.

(ii) Express y in terms of a, b, c and h.

(b) Suppose $h = 3$ and (E) is consistent.

(i) Show that $3a - 4b + c = 0$ when (E) has infinitely many solutions.

(ii) Solve the system of equations $\begin{cases} 3x + 3y + z = 2 \\ 9x + 9y + z = 1 \\ 27x + 27y + z = -2 \\ 3x + 6y - 2z = 0 \end{cases}$.

(9 marks)

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7. (a) Evaluate $\int (4-\sqrt{t+9})^2 dt$.

(b) Consider the curve $\Gamma : y = x^2 - 8x + 7$ where $1 \le x \le 4$. Let R be the region bounded by Γ, the straight line $y = -5$ and the two axes. Find the volume of solid of revolution generated by revolving R about the y-axis.

(6 marks)

8. It is given that the equation of a curve is $y=\dfrac{x+1}{e^{2x}}$.

(a) Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$.

(b) (i) Find all the extreme point(s) of the curve and state whether it is a maximum point or a minimum point.

(ii) Find the point(s) of inflexion of the curve.

(7 marks)

Section B (50 marks)

9. Let $g(x) = \cos^2 x \cos 2x$.

(a) Prove that $\int g(x)\,dx = \dfrac{\sin 2x \cos^2 x}{2} + \dfrac{1}{2}\int \sin^2 2x\,dx$. (2 marks)

(b) Evaluate $\int_0^{\pi} g(x)\,dx$. (3 marks)

(c) Using integration by substitution, evaluate $\int_0^{\pi} xg(x)\,dx$. (4 marks)

(d) (i) Show that $xg(x)$ is an odd function.

(ii) Evaluate $\int_{-\pi}^{2\pi} xg(x)\,dx$.

(4 marks)

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10. In Figure 1, suppose $\overrightarrow{OA}=\mathbf{a}$, $\overrightarrow{OB}=\mathbf{b}$ and $\overrightarrow{OC}=\mathbf{c}$. BO is produced to meet AC at M. It is given that $BO:OM=r:(1-r)$ and $AM:MC=s:(1-s)$, where $0<r<1$ and $0<s<1$.

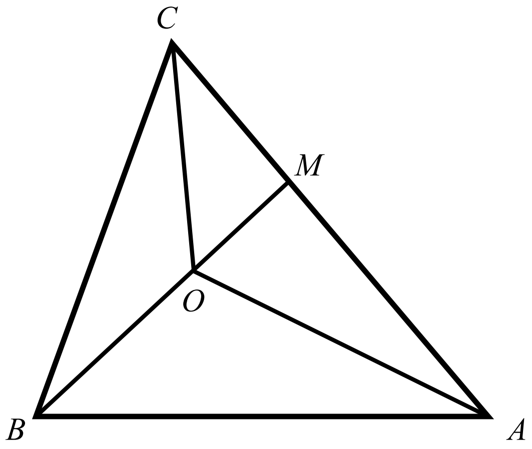

Figure 1

(a) (i) Express $\overrightarrow{BM}$ in terms of r and $\mathbf{b}$.

(ii) Express $\overrightarrow{BM}$ in terms of s, $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

(3 marks)

(b) Let $\mathbf{c}=x\mathbf{a}+y\mathbf{b}$. Show that $x=\dfrac{s-1}{s}$ and $y=\dfrac{r-1}{rs}$. (3 marks)

(c) Suppose $|\mathbf{a}|=3$, $|\mathbf{b}|=2$ and $\angle AOB=120°$. If O is the orthocentre of ΔABC, find the values of r and s. (6 marks)

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11. Let $A = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}$. Denote the 2×2 identity matrix by I.

(a) Using mathematical induction, prove that $A^n = 2^n I + 2^{n-1} n \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ for all positive integers n. (4 marks)

(b) Let $B = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}$.

(i) Define $P = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Evaluate $P^{-1}BP$.

(ii) Prove that $B^n = 2^n I + 2^{n-1} n \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$ for any positive integer n.

(iii) Does there exist a positive integer m such that $\left| A^m - B^m \right| = m^2$? Explain your answer.

(9 marks)

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12. (a) In Figure 2, the shaded region is bounded by the circle $x^2 + y^2 = r^2$, the y-axis and the straight line $y = r - h$, where $0 \le h \le 2r$. Show that the volume of the solid generated by revolving the shaded region about the y-axis is $\pi r h^2 - \dfrac{\pi h^3}{3}$.

(4 marks)

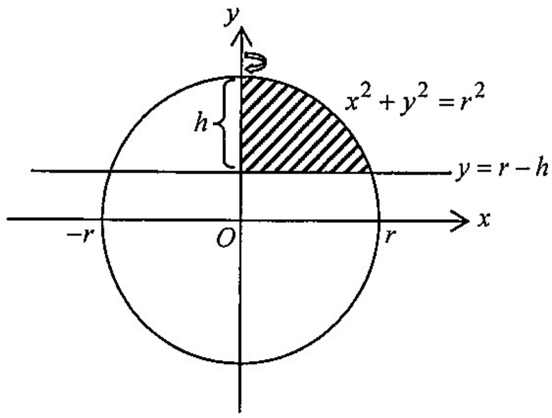

Figure 2

(b)

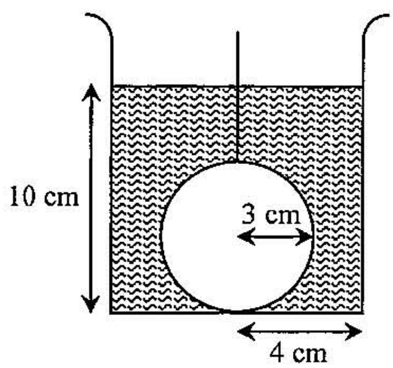

Figure 3

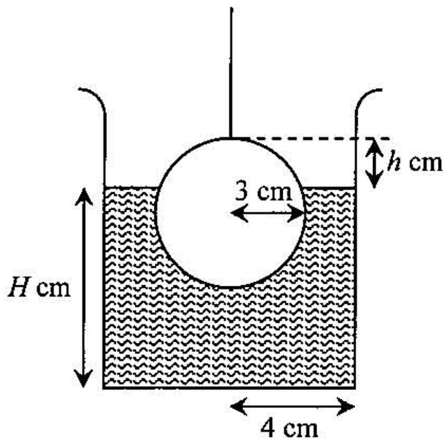

Figure 4

A metal sphere of radius 3 cm, with a thin string attached, is placed inside a circular cylindrical container of base radius 4 cm. Water is poured into the container until the depth of the water is 10 cm (see Figure 3). The sphere is then being pulled vertically out of the water. Let H cm and h cm be the depth of the water and the distance between the top of the sphere and the water surface respectively (see Figure 4).

(i) Prove that $H = \dfrac{1}{48}(h^3 - 9h^2 + 480)$.

(ii) The sphere is being pulled at a constant speed of $\dfrac{1}{4}$ cm s^{-1}. At the instant when $h = 3$, find the rate of change of

(1) the depth of the water,

(2) the distance between the top of the sphere and the water surface.

(8 marks)

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