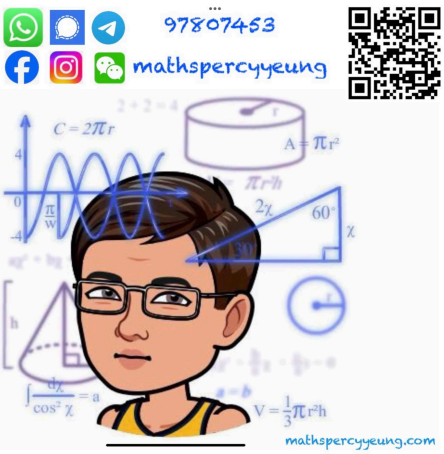

2021-2022 S6
MOCK EXAM
MATH EP
M2

2021 – 2022
S6 Mock Examination

MATHEMATICS Extended Part

Module 2 (Algebra and Calculus)

Question–Answer Book

19th January, 2022
8:15 am – 10:45 am (2 hours 30 minutes)
This paper must be answered in English

INSTRUCTIONS

1. Write your name, class and class number in the spaces provided on this cover.

2. This paper consists of TWO sections, A and B.

3. Attempt ALL questions in this paper. Write your answers in the spaces provided in this Question – Answer Book. Do not write in the margins. Answers written in the margins will not be marked.

4. Unless otherwise specified, all working must be clearly shown.

5. Unless otherwise specified, numerical answers must be exact.

6. The diagrams in this paper are not necessarily drawn to scale.

Sections	Marks
A Total	**/50**
B Total	**/50**
TOTAL	**/100**

FORMULAS FOR REFERENCE

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$	$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$
$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$	$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$
$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$	$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$
$2\sin A \cos B = \sin(A+B) + \sin(A-B)$	$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$
$2\cos A \cos B = \cos(A+B) + \cos(A-B)$	
$2\sin A \sin B = \cos(A-B) - \cos(A+B)$	

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Section A (50 marks)

1. Find $\frac{d}{dx}\left(\frac{1}{\sqrt{x}+2}\right)$ from first principles. (4 marks)

Answers written in the margins will not be marked

2. By using integration by parts, find $\int \frac{xe^x}{(1+x)^2}\,dx$. (4 marks)

Answers written in the margins will not be marked

3. The coefficient of x in the expansion of $(1+x)^2(1-3x)^n$ is -19, where n is a positive integer. Find the value of n and the coefficient of x^3. (5 marks)

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4. Let $P = \begin{pmatrix} 3 & -3 & 8 \\ 4 & -7 & 0 \end{pmatrix}$ and $Q = \begin{pmatrix} -3 & -1 \\ 0 & -8 \\ -6 & -9 \end{pmatrix}$.

(a) Find PQ.

(b) Without finding the inverse of PQ, show that PQ is invertible.

(c) Find $(PQ)^{-1}$.

(5 marks)

Answers written in the margins will not be marked

5. Consider the following system of linear equations in x, y and z.

$$(E): \begin{cases} x + ay - 4z = 1 \\ 2x + (2a+6)y + (3ab-8)z = -15-b \\ 3x + (3a-11)y + (11b-12)z = 36 \end{cases}$$

It is given that (E) has infinitely many solutions.

(a) Find the values of a and b.

(b) Solve (E).

(5 marks)

Answers written in the margins will not be marked

6. **(a)** By considering $3\theta = 2\theta + \theta$, express $\sin 3\theta$ in the form of $A\sin\theta + B\sin^3\theta$, where A and B are constants.

(b) Using **(a)**, solve the equation $8x^3 - 6x + 1 = 0$.

(7 marks)

Answers written in the margins will not be marked

7. It is given that $OACB$ is a parallelogram. $\overrightarrow{OA} = 5\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\overrightarrow{OB} = 2x\mathbf{i} + x\mathbf{j} + p\mathbf{k}$, where p is a non-zero constant.

(a) Find the area of $OACB$ in terms of x and p.

(b) If the area of $OACB$ is minimized,

(i) show that $x = \frac{11}{14}p$;

(ii) is $\angle AOB$ equal to 90°? Explain your answer.

(6 marks)

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8. **(a)** Prove, by mathematical induction, that

$$\frac{1^2+1+1}{1\times 2}+\frac{2^2+2+1}{2\times 3}+\cdots+\frac{n^2+n+1}{n(n+1)}=n+1-\frac{1}{n+1}$$

for all positive integers n.

(b) Hence, evaluate $\sum_{k=10}^{100}\frac{k^2+k+1}{k^2+k}$.

(7 marks)

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9. A container in the form of an inverted right circular cone is held vertically. The height and the base radius of the container are 15 cm and 20 cm respectively.

(a) Let V cm^3 and A cm^2 be the volume of water in the container and the wet surface area of the container respectively and h be the depth of water in the container. Prove that

$$V = \frac{16\pi}{27}h^3 \text{ and } A = \frac{20\pi}{9}h^2.$$

(b) Water is flowing out of the container at a constant rate of $9\pi \text{ cm}^3\text{ s}^{-1}$. Find the rate of change of the wet surface area of the container when the volume of water in the container is $432\pi \text{ cm}^3$.

(7 marks)

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Section B (50 marks)

10. Define $f(x)=2-\frac{1}{x-5}+\frac{1}{x-1}$, for all $x\neq 1$ and $x\neq 5$. Denote the graph of $y=f(x)$ as G.

(a) **(i)** Write down the asymptotes of G.

(ii) Find the x- and y-intercepts of G.

(iii) Write down $f'(x)$ for $x\neq 1$ and $x\neq 5$.

(iv) For G, find all the extreme points.

(7 marks)

(b) Sketch G. (2 marks)

(c) Let S be the area bounded by G and the lines $x=6$, $x=k$ and $y=2$, where $k>6$. Prove that $S<\ln 5$. (3 marks)

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11. Let $A=\begin{pmatrix}4 & 1\\ 1 & 4\end{pmatrix}$, $B=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$, $C=\begin{pmatrix}x & 0\\ 0 & y\end{pmatrix}$ and $I=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$, where $0<\theta<\frac{\pi}{2}$.

(a) Prove that $B^{-1}AB=\begin{pmatrix}4+\sin 2\theta & \cos 2\theta\\ \cos 2\theta & 4-\sin 2\theta\end{pmatrix}$. (3 marks)

(b) Furthermore, $B^{-1}AB=C$.

(i) Find θ, x and y.

(ii) If $C^n=\lambda C+\mu I$ where n is a positive integer, and λ and μ are constants, express λ and μ in terms of n.

(iii) Using the results of **(b)(i)** and **(b)(ii)**, express A^n in terms of n.

(10 marks)

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12. **(a)** Let $f(x)$ be a continuous function defined on the interval $[0,\ a]$, where a is a positive constant. Prove that $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$. (3 marks)

(b) Let θ be a positive constant where $0 < \theta < \frac{\pi}{2}$.

(i) Prove that $\int_0^\theta \ln(1+\tan\theta\tan x)\,dx = \int_0^\theta \ln\left(\frac{1+\tan^2\theta}{1+\tan\theta\tan x}\right)dx$

(ii) Hence, prove that $\int_0^\theta \ln(1+\tan\theta\tan x)\,dx = \theta\ln\sec\theta$.

(6 marks)

(c) Evaluate $\int_0^{\frac{\pi}{3}} \frac{x\sec^2 x}{1+\sqrt{3}\tan x}\,dx$. (4 marks)

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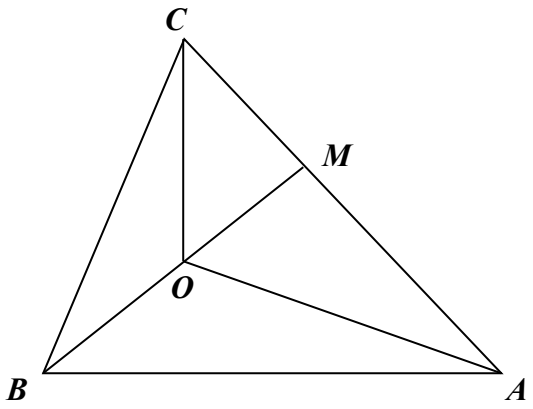
13. In the figure, suppose $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and $\overrightarrow{OC} = \mathbf{c}$. BO is produced to M on AC. It is given that $BO : OM = r : (1-r)$ and $AM : MC = s : (1-s)$, where $0 < r < 1$ and $0 < s < 1$.



(a) **(i)** Express $\overrightarrow{BM}$ in terms of r and **b.**

(ii) Express $\overrightarrow{BM}$ in terms of s, **a**, **b** and **c**.

(3 marks)

(b) Let $\mathbf{c} = x\mathbf{a} + y\mathbf{b}$. Show that $x = \dfrac{s-1}{s}$ and $y = \dfrac{r-1}{rs}$. (3 marks)

(c) Suppose $|\mathbf{a}| = 3$, $|\mathbf{b}| = 2$ and $\angle AOB = 120°$. If O is the orthocentre of ΔABC, find the values of r and s. (6 marks)

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