2020-21
MATH EP
F.6 M2

2020 - 2021 Mock Examination

Form 6 MATHEMATICS
Extended Part
Module 2 (Algebra and Calculus)

Question–Answer Book

13th January, 2021. (Wednesday)

8:15 am – 10:45 am (2.5 hours)

This paper must be answered in English.

INSTRUCTIONS

1. After the announcement of the start of the examination, you should first write your name, class and class number in the spaces provided on this cover.
2. This paper consists of Section A and Section B.
3. Answer ALL questions. Write your answers in the spaces provided in this Question-Answer Book.
4. Graph paper and supplementary answer sheets will be supplied on request. Write your name, class, class number and mark the question number box on each sheet.
5. Unless otherwise specified, all working must be clearly shown.
6. Unless otherwise specified, numerical answers must be exact.
7. The diagrams in this paper are not necessarily drawn to scale.

	Marks
Section A	/ 50
Section B	/ 50
Grand Total	%

FORMULAS FOR REFERENCE

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$	$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$
$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$	$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$
$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$	$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$
$2\sin A \cos B = \sin(A+B) + \sin(A-B)$	$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$
$2\cos A \cos B = \cos(A+B) + \cos(A-B)$	
$2\sin A \sin B = \cos(A-B) - \cos(A+B)$	

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Section A (50 marks)

1. Find $\frac{d}{dx}(\sin^2 x)$ from first principles. (4 marks)

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2. Let n be a positive integer. In the expansion of $(1+3x)^n\left(x-\dfrac{4}{x}\right)^2$, the constant term is 5 176.

(a) Find the value of n .

(b) Find the coefficient of x^2 in the expansion.

(5 marks)

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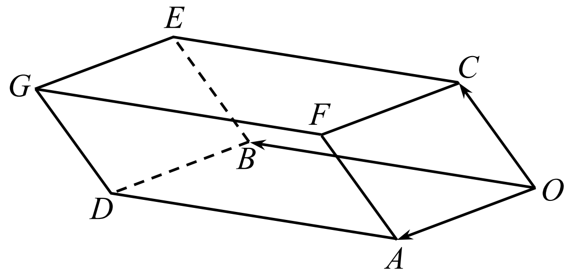
3. The figure below shows a parallelepiped *OADBECFG*. Let $\overrightarrow{OA} = 2\mathbf{i} + 3\mathbf{k}$, $\overrightarrow{OB} = 3\mathbf{i} - \mathbf{j}$ and $\overrightarrow{OC} = -3\mathbf{j} + 4\mathbf{k}$.



(a) Find the volume of the parallelepiped *OADBECFG* .

(b) Find the acute angle between the plane *OADB* and the line *OF* , correct to the nearest degree.

(6 marks)

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4. Find the equation of the tangent to the curve $xe^y - y^3\cos x = 5$ at the point where the curve cuts the x-axis. (5 marks)

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5. (a) Prove that $\tan\left(\dfrac{x+y}{2}\right)=\dfrac{\sin x+\sin y}{\cos x+\cos y}$.

(b) Let $0<\theta<\pi$ and $\theta\neq\dfrac{\pi}{2}$.

(i) Prove that $\dfrac{\tan 2\left(\dfrac{\theta}{2}-\dfrac{\pi}{4}\right)}{\tan\left(\dfrac{\theta}{2}-\dfrac{\pi}{4}\right)}=\csc\theta+1$.

(ii) Solve the equation $\dfrac{\tan 2\left(\dfrac{\theta}{2}-\dfrac{\pi}{4}\right)}{\tan\left(\dfrac{\theta}{2}-\dfrac{\pi}{4}\right)}=3$.

(7 marks)

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6. (a) Using mathematical induction, prove that $\sum_{k=n}^{2n} k^2 = \frac{n(n+1)(14n+1)}{6}$ for all positive integers n.

(b) Using (a), evaluate $\sum_{k=25}^{100} k^2$.

(8 marks)

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7. (a) Evaluate $\int e^{2x}\cos 2x\,\mathrm{d}x$.

(b) Denote the graph of $y=e^{x}\cos x$ by G . Let R be the region bounded by G , the x-axis, the y-axis and the vertical line $x=\frac{\pi}{4}$. Find the volume of the solid of revolution generated by revolving R about the x-axis.

(7 marks)

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8. Define $P=\begin{pmatrix}7 & 3\\ -14 & -6\end{pmatrix}$ and $Q=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$. Let $M=\begin{pmatrix}1 & a\\ b & c\end{pmatrix}$ such that $|M|=1$ and $PM=MQ$, where a , b and c are real numbers.

(a) Find a , b and c .

(b) Define $R=\begin{pmatrix}-6 & -3\\ 14 & 7\end{pmatrix}$.

(i) Evaluate $M^{-1}RM$.

(ii) Using the result of (b) (i), prove that $(\alpha P+\beta R)^{999}=\alpha^{999}P+\beta^{999}R$ for any real numbers α and β .

(8 marks)

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Section B (50 marks)

9. A producer designs a tank *ABCDEFGH* (as shown in Figure 1) in the shape of a right prism whose base *ABCD* is a trapezium. $AB = BC = CD = 2$ m and $AF = 5$ m. Let V m^3 be the capacity of the tank and $\angle BAD = \angle CDA = \theta$, where $0 < \theta < \frac{\pi}{2}$.

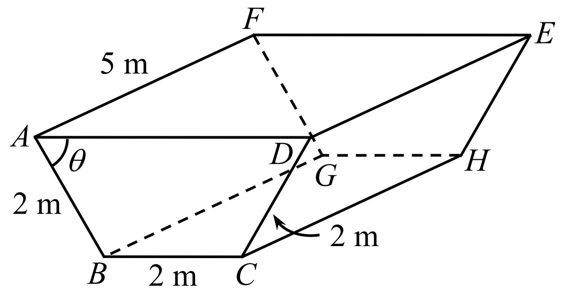

Figure 1

(a) Show that $V = 20 \sin \theta \, (\cos \theta + 1)$. (2 marks)

(b) When the capacity of the tank is maximum, find θ. (4 marks)

(c) The producer makes a tank based on θ obtained in (b). Initially, the tank is empty and the face *BCHG* is placed on a horizontal table. Now water is poured into the tank.

(i) Suppose the volume of water in the tank increases at a constant rate of 5 m^3/h. Find the rate of increase of the depth of water when the depth of water is $\frac{\sqrt{3}}{2}$ m.

(ii) Suppose the depth of water in the tank increases at a constant rate. The producer claims that the rate of increase of the volume of water in the tank varies directly as the area of the water surface. Do you agree? Explain your answer.

(6 marks)

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10. Let $a>0$ and $f(x)$ be a continuous function. It is given that $f(a-x)=f(x)$ for all real values of x.

(a) Prove that $\int_0^a x\,f(x)\,dx=\int_0^a (a-x)\,f(x)\,dx$.

Hence, prove that $\int_0^a x\,f(x)\,dx=\dfrac{a}{2}\int_0^a f(x)\,dx$. (3 marks)

(b) Show that $\int_0^2 \dfrac{1}{x^2-2x+4}\,dx=\dfrac{\sqrt{3}\,\pi}{9}$. (3 marks)

(c) (i) Prove that $\int_0^2 \dfrac{x}{x^2-2x+4}\,dx=\dfrac{1}{2}\int_0^{\frac{\pi}{2}} \dfrac{\sin 2\theta}{1-\cos\theta+\cos^2\theta}\,d\theta$.

(ii) Using (a) and (b), or otherwise, evaluate $\int_0^{\frac{\pi}{2}} \dfrac{2\tan\theta}{2-\sec\theta+\tan^2\theta}\,d\theta$.

(7 marks)

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11. (a) Consider the system of linear equations in real variables x , y , z

$$(E):\begin{cases} x-\ \ y+\ \ z = 0 \\ x+k^2y+(3k-8)z = 3a \\ 3x-3y+kz = a+1 \end{cases},$$

where a and k are real numbers.

(i) Assume that (E) has a unique solution.

(1) Find the range of values of k .

(2) Solve (E) for $k=1$.

(ii) Assume that $k=3$ and (E) is consistent.

(1) Find a .

(2) Solve (E) .

(9 marks)

(b) Consider the system of equations in real variables x , y , z

$$(F):\begin{cases} x-y+z=0 \\ x+9y+z=3c \\ 6x-2(3y+1)+6z=2c \\ 10x+10y+z^2=-27 \end{cases},$$

where c is a constant.

Assume that (F) is consistent. Solve (F) .

(4 marks)

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12. Let $\overrightarrow{OA} = 6\mathbf{i} - 4\mathbf{j} + 6\mathbf{k}$, $\overrightarrow{OB} = 5\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}$ and $\overrightarrow{OC} = -\mathbf{i} + 12\mathbf{k}$. It is given that D is a point lying on BC such that AD is perpendicular to BC.

(a) Find $\overrightarrow{AD}$. (4 marks)

(b) Denote the plane which contains A, B and C by Π. Let $\overrightarrow{OE} = 24\mathbf{i} + m\mathbf{j} + n\mathbf{k}$, where m and n are constants. It is given that A is the projection of E on Π.

(i) Find $\overrightarrow{AB} \times \overrightarrow{AC}$.

(ii) Find the values of m and n.

(iii) Is $\overrightarrow{ED}$ perpendicular to $\overrightarrow{BC}$? Explain your answer.

(iv) Find the angle between ΔBCE and Π.

(8 marks)

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