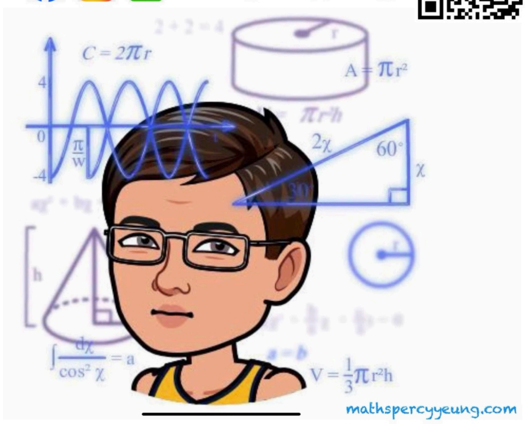
2019-20
F.6 MATH
EP M2



2019 - 2020 Mock Examination

MATHEMATICS Extended Part
Module 2 (Algebra and Calculus)

Question–Answer Book

15th January, 2020. (Wednesday)

8:15 am – 10:45 am (2.5 hours)

This paper must be answered in English.

Section	Marks
A	**/ 50**
B	**/ 50**
TOTAL	**%**

INSTRUCTIONS

1. After the announcement of the start of the examination, you should first write your name, class and class number in the spaces provided on this cover.
2. This paper consists of Section A and Section B.
3. Answer ALL questions. Write your answers in the spaces provided in this Question-Answer Book.
4. Graph paper and supplementary answer sheets will be supplied on request. Write your name, class, class number and mark the question number box on each sheet.
5. Unless otherwise specified, all working must be clearly shown.
6. Unless otherwise specified, numerical answers must be exact.
7. In this paper, vectors may be represented by bold-type letters such as **u**, but candidates are expected to use appropriate symbols such as $\vec{u}$ in their working.
8. The diagrams in this paper are not necessarily drawn to scale.

FORMULAS FOR REFERENCE

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$	$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$
$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$	$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$
$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$	$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$
$2\sin A \cos B = \sin(A+B) + \sin(A-B)$	$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$
$2\cos A \cos B = \cos(A+B) + \cos(A-B)$	
$2\sin A \sin B = \cos(A-B) - \cos(A+B)$	

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Section A (50 marks)

1. In the expansion of $(1+x)^n(1-2x)^{n+1}$, the coefficient of x^2 is 6.

(a) Find the value of n.

(b) Find the coefficient of the term with the highest power of x. (6 marks)

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2. Find $\lim_{n\to\infty}\left(1-\frac{4}{n}\right)^{2n+1}$. (4 marks)

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3. (a) Prove by mathematical induction that

$$1-\frac{3}{1\times 4}-\frac{3}{4\times 7}-\frac{3}{7\times 10}-\ldots-\frac{3}{(3n-2)(3n+1)}=\frac{1}{3n+1}$$

for all positive integers n.

(b) Hence, find the sum $\frac{2}{1\times 4}+\frac{2}{4\times 7}+\frac{2}{7\times 10}+\ldots+\frac{2}{133\times 136}$. (7 marks)

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4. Let C be the curve $x = y^3 - 3y^2$.

(a) Express $\frac{dy}{dx}$ in terms of y.

(b) Find the equation of the tangent to C if the slope of the tangent is $-\frac{1}{3}$. (5 marks)

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5. (a) Using integration by parts, or otherwise, find $\int xe^x \, dx$ and $\int x^2 e^x \, dx$.

(b) Show that $\int_0^x f(t)\, dt = -\int_0^{-x} f(-t)\, dt$. Hence, deduce that $f(x)$ is an even function, then the function $g(x) = \int_0^x f(t)\, dt$ is an odd function. (7 marks)

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6. (a) Prove that $\tan 2\theta = \dfrac{2\tan\theta}{2-\sec^2\theta}$.

(b) Using (a), prove that $\tan 8x = \dfrac{8\tan x}{(2-\sec^2 4x)(2-\sec^2 2x)(2-\sec^2 x)}$. (5 marks)

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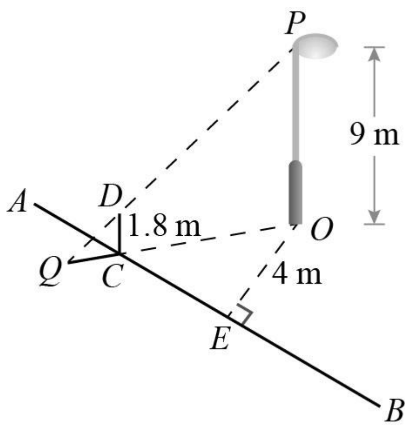
7. In the figure, a lamp post OP is 9 m tall and it is standing 4 m away from a straight road AB. A man CD is walking on the road from A towards E with a constant speed of 1 m/s. His eye level is 1.8 m and he casts a shadow CQ on the ground. Let $CQ = x$ m and $OC = y$ m.



(a) (i) Show that $\frac{dy}{dt} = 4\frac{dx}{dt}$.

(ii) Let $CE = s$ m. Find the rate of change of the length of the shadow when $y = 5$.

(4 marks)

(b) Find the rate of change of the angle of elevation of the top of the lamp post from the man when $y = 5$. (Give your answer correct to 3 significant figures.) (4 marks)

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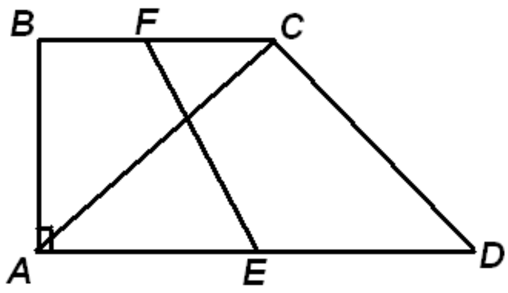
8. In the figure, $ABCD$ is a trapezium with $BC \,//\, AD$ and $\angle BAD = 90°$. It is given that $AB = BC$ and $AD = kBC$, where k is a real number. E and F are the mid-points of AD and BC respectively. Let $\overrightarrow{BF} = \mathbf{f}$ and $\overrightarrow{BA} = \mathbf{a}$.



(a) Express $\overrightarrow{EF}$ in terms of k, $\mathbf{f}$ and $\mathbf{a}$.

(b) Find the value of k such that $AC \perp EF$. (8 marks)

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Section B (50 marks)

9. (a) Consider the system of linear equations in x, y and z:

$$(E):\begin{cases} \qquad my - 2z = -m \\ x - y + (m-1)z = m+1 \\ x + y - 4z = n \end{cases}, \text{ where } m \text{ and } n \text{ are real numbers.}$$

(i) Assume that (E) has a unique solution.

(1) Prove that $m \neq -4$ and $m \neq 1$.

(2) Solve (E) for $n = 0$.

(ii) Assume that $m = 1$ and (E) is consistent.

(1) Find n.

(2) Solve (E). (9 marks)

(b) It is known that $(T):\begin{cases} y-2z=-1 \\ x-y=2 \\ x+y-4z=n \\ x^2+y^2+z^2=83 \end{cases}$ is consistent. Solve (T). (3 marks)

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10. (a) Find $\int \frac{dx}{x^2 - x + \frac{1}{2}}$. (3 marks)

(b) Let $f(x)$ be a continuous function defined on the interval $[-\lambda, \lambda]$, where λ is a positive constant. If $f(x) = f(\lambda - x)$, prove that $\int_0^\lambda x\, f(x)dx = \frac{\lambda}{2}\int_0^\lambda f(x)dx$. (3 marks)

(c) Consider the curve $G: y = \sqrt{\frac{x \sin x \cos x}{\sin^4 x + \cos^4 x}}$, where $0 \le x \le \frac{\pi}{2}$. Let R be the region bounded by G and the x-axis. Find the volume of the solid of revolution generated by revolving R about the x-axis. (6 marks)

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11. For any matrix $M=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, define $\operatorname{tr}(M)=a+d$. Let A and B be 2×2 matrices such that $BAB^{-1}=\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$.

(a) (i) For any matrix $N=\begin{pmatrix} e & f \\ g & h \end{pmatrix}$, prove that $\operatorname{tr}(MN)=\operatorname{tr}(NM)$.

(ii) Find the value of $|A|$.

(iii) Using the result of (a)(i), show that $\operatorname{tr}(A)=4$.

(6 marks)

(b) Let $C=\begin{pmatrix} p & q \\ r & s \end{pmatrix}$. It is given that $C\begin{pmatrix} x \\ y \end{pmatrix}=\lambda_1\begin{pmatrix} x \\ y \end{pmatrix}$ and $C\begin{pmatrix} x \\ y \end{pmatrix}=\lambda_2\begin{pmatrix} x \\ y \end{pmatrix}$ for some non-zero matrices $\begin{pmatrix} x \\ y \end{pmatrix}$ and distinct scalars λ_1 and λ_2.

(i) Prove that $\begin{vmatrix} p-\lambda_1 & q \\ r & s-\lambda_1 \end{vmatrix}=0$ and $\begin{vmatrix} p-\lambda_2 & q \\ r & s-\lambda_2 \end{vmatrix}=0$.

(ii) Prove that λ_1 and λ_2 are the roots of the equation $\lambda^2-\operatorname{tr}(C)\cdot\lambda+|C|=0$.

(5 marks)

(c) Find the two values of λ such that $A\begin{pmatrix} x \\ y \end{pmatrix}=\lambda\begin{pmatrix} x \\ y \end{pmatrix}$ for some non-zero matrices $\begin{pmatrix} x \\ y \end{pmatrix}$.

(2 marks)

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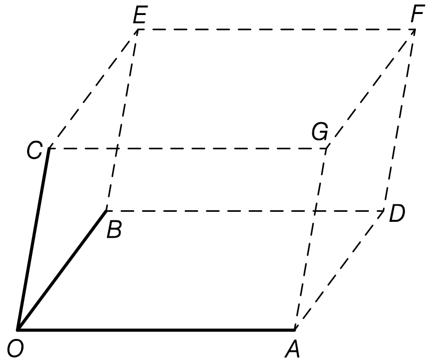

Figure 1

(a) In Figure 1, A, B and C are three distinct points with $\overrightarrow{OA} = a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k}$, $\overrightarrow{OB} = a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k}$ and $\overrightarrow{OC} = a_3\mathbf{i} + b_3\mathbf{j} + c_3\mathbf{k}$. By considering the volume of the parallelepiped $OADBECGF$ which is formed by $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$, show that $\overrightarrow{OA}, \overrightarrow{OB}$ and $\overrightarrow{OC}$ are coplanar if and only if $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$. (2 marks)

(b) Let $\overrightarrow{OP} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\overrightarrow{OQ} = \mathbf{i} - \mathbf{j}$ and $\overrightarrow{OR} = \mathbf{i} + 2\mathbf{j} + 5\mathbf{k}$

(i) Let $\mathbf{e}_1$ and $\mathbf{e}_2$ be the unit vectors of $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ respectively.
Express $\mathbf{e}_1$ and $\mathbf{e}_2$ in terms of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$.

(ii) Let $\overrightarrow{OS} = \left[\overrightarrow{OR} \cdot (\mathbf{e}_1 \times \mathbf{e}_2)\right](\mathbf{e}_1 \times \mathbf{e}_2)$. Find $\overrightarrow{OS}$ and hence, show that $\overrightarrow{RS}$, $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ are coplanar.

(iii) Show that $\overrightarrow{OR}$ can be expressed as the sum of two vectors $\mathbf{u}$ and $\mathbf{v}$, where $\mathbf{u}$ is perpendicular to $\overrightarrow{OP}$ and $\overrightarrow{OQ}$, and $\mathbf{v}$ is coplanar with $\overrightarrow{OP}$ and $\overrightarrow{OQ}$. (11 marks)

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