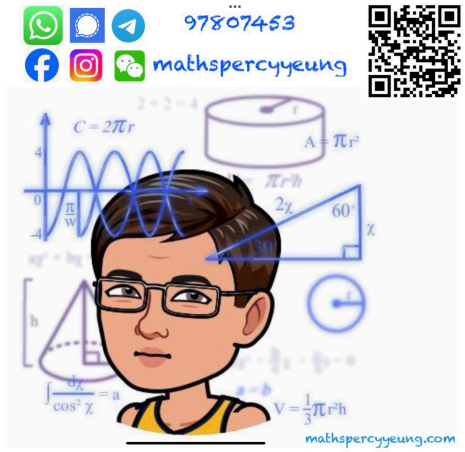
**2018-19**
**MATH EP**
**M2**




**2018 - 2019 Mock Examination**

# MATHEMATICS Extended Part Module 2 (Algebra and Calculus)

## Question–Answer Book

15th January, 2019. (Tuesday)

8:15 am – 10:45 am (2.5 hours)

This paper must be answered in English.

| Section | Marks |
|---|---|
| **A** | **/ 50** |
| **B** | **/ 50** |
| **TOTAL** | **%** |

### INSTRUCTIONS

1. After the announcement of the start of the examination, you should first write your name, class and class number in the spaces provided on this cover.
2. This paper consists of Section A and Section B.
3. Answer ALL questions. Write your answers in the spaces provided in this Question-Answer Book.
4. Graph paper and supplementary answer sheets will be supplied on request. Write your name, class, class number and mark the question number box on each sheet.
5. Unless otherwise specified, all working must be clearly shown.
6. Unless otherwise specified, numerical answers must be exact.
7. In this paper, vectors may be represented by bold-type letters such as **u**, but candidates are expected to use appropriate symbols such as $\vec{u}$ in their working.
8. The diagrams in this paper are not necessarily drawn to scale.

**FORMULAS FOR REFERENCE**

| | |
|---|---|
| $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$ | $\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$ |
| $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$ | $\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$ |
| $\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$ | $\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$ |
| $2\sin A \cos B = \sin(A+B) + \sin(A-B)$ | $\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$ |
| $2\cos A \cos B = \cos(A+B) + \cos(A-B)$ | |
| $2\sin A \sin B = \cos(A-B) - \cos(A+B)$ | |

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**Section A** (50 marks)

**1.** Find $\frac{d}{dx}(\ln 2x)$ from first principles. (4 marks)



Answers written in the margins will not be marked.

**2.** In the expansion of $\left(2-\frac{1}{x}\right)^2(1-3x)^n$, the constant term is 154. Find the value of $n$ and the coefficient of $x^2$. (4 marks)



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**3.** Denote the graph of $e^y = \cos y + xe^{2x-y}$ by $C$.
Find the equation of the normal to $C$ at $y = 0$. (5 marks)



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**4.** **(a)** Prove that $\csc 2x = \cot x - \cot 2x$ .

**(b)** Let $f(x) = \dfrac{3 + 2\cot 4x}{\csc 8x + \cot 8x}$.

**(i)** Express $f(x)$ in the form of $A\tan Bx + C$, where $A$, $B$ and $C$ are constants.

**(ii)** Solve the equation $f(x) = 5$, where $0 < x < \dfrac{\pi}{2}$.

(5 marks)



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**5.** In Figure 1, a helicopter takes off at $H$ with a constant speed of 8 m/s. The helicopter flies along a straight path which make an angle of $\frac{\pi}{6}$ to the vertical. $A$ is a point 500 m from $H$ which lies on the same horizontal ground of $H$. Let $\theta$ be the angle of elevation of the helicopter from $A$ at $t$ seconds after the take-off. Assume that the flight path of the helicopter and $A$ lies on the same vertical plane. .

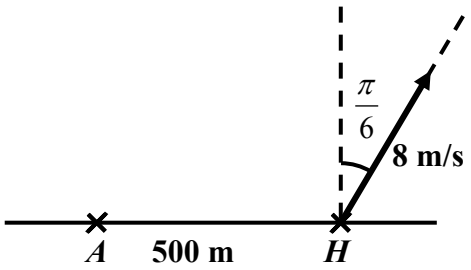


**Figure 1**

**(a)** Express $\tan\theta$ in terms of $t$.

**(b)** Find the rate of change of $\theta$ when $\theta = \frac{\pi}{6}$ .

(5 marks)



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**6.** **(a)** Find $\int x^2 e^{-x} dx$.

**(b)** Figure 2 shows the shaded region bounded by the curve $y = xe^{-\frac{x}{2}}$, the line $x = 3$ and the $x$-axis. Find the volume of solid generated by revolving the region about the $x$-axis.

(6 marks)

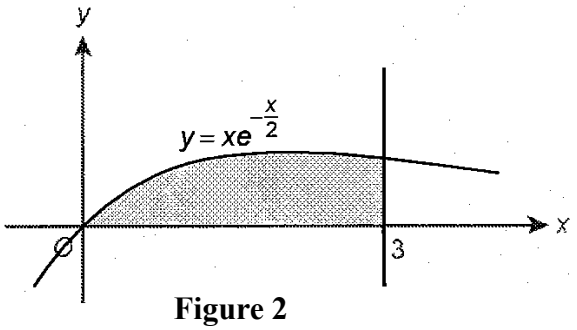


**Figure 2**



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**7.** **(a)** Using Mathematical induction, prove that $\sum_{k=1}^{n} \frac{(-1)^{k+1}(2k+1)}{k(k+1)} = 1 - \frac{(-1)^n}{n+1}$ for all positive integers $n$.

**(b)** Using **(a)**, evaluate $\sum_{k=6}^{201} \frac{(-1)^k(2k+1)}{3k(k+1)}$.

(6 marks)



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**8.** Let $\overrightarrow{OA} = 8\mathbf{i} + 4\mathbf{j}$, $\overrightarrow{OB} = -\mathbf{i} + 7\mathbf{j}$. $M$ and $N$ are two points on $OA$ and $AB$ respectively such that $OM : MA = r : 1$ and $BN : NA = s : 1$, where $r$ and $s$ are real numbers.

**(a)** If $ON \perp AB$, find the value of $s$.

**(b)** Peter claims that if $r = \frac{1}{3}$, then $O$, $B$, $M$ and $N$ are concyclic. Do you agree? Explain your answer.

(7 marks)



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**9.** **(a)** Solve the following system of linear equations:

$$\begin{cases} x - y + 3z = 12 \\ 2x - y - z = -5 \end{cases}$$

**(b)** **(i)** Using **(a)**, or otherwise, solve the following system of linear equations:

$$(E): \begin{cases} x - y + 3z = 12 \\ 2x - y - z = -5 \\ 4x - 3y + \lambda z = 19 \end{cases}$$

, where $\lambda$ is a constant.

**(ii)** Suppose $\lambda = 5$. Calvin claims that there is at least one set of solution such that $x$, $y$ and $z$ are all positive even numbers. Do you agree? Explain your answer.

(8 marks)



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**Section B** (50 marks)

**10.** Define $f(x) = 2 - \frac{1}{x-5} + \frac{1}{x-1}$, for all $x \neq 1$ and $x \neq 5$. Denote the graph of $y = f(x)$ as $G$.

**(a)** **(i)** Write down the asymptotes of $G$.

**(ii)** Find the $x$- and $y$-intercepts of $G$.

**(iii)** Write down $f'(x)$ for $x \neq 1$ and $x \neq 5$.

**(iv)** For $G$, find all the extreme points.

(7 marks)

**(b)** Sketch $G$. (2 marks)

**(c)** Let $S$ be the area bounded by $G$ and the lines $x = 6$, $x = k$ and $y = 2$, where $k > 6$.

Prove that $S < \ln 5$. (3 marks)



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**11.** Let $M = \begin{pmatrix} 3+x & -y \\ -x & 3+y \end{pmatrix}$ and $P = \begin{pmatrix} y & 1 \\ x & -1 \end{pmatrix}$, where $x$ and $y$ are real numbers such that $xy > 0$.

**(a)** Show that $P$ is an invertible matrix. (2 marks)

**(b)** Evaluate $P^{-1}MP$.
Hence find $P^{-1}M^n P$ in terms of $x$, $y$ and $n$, where $n$ is a positive integer. (5 marks)

**(c)** Suppose that $A = \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix}$ and $n$ is a positive integer. Using the result of **(b)**, find $A^{2n}$ in terms of $n$. (4 marks)



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**12.** **(a)** **(i)** Show that $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$.

**(ii)** Hence show that $\int_a^b f(x)dx = \frac{1}{2}\int_a^b [f(x)+f(a+b-x)]dx$.

(4 marks)

**(b)** **(i)** Show that $\sin x + \cos x \equiv \sqrt{2}\sin\left(x+\frac{\pi}{4}\right)$.

**(ii)** Hence evaluate $\int_0^{\frac{\pi}{2}} \frac{dx}{\sin x + \cos x}$.

(6 marks)

**(c)** Using the results of **(a)** and **(b)**, or otherwise, evaluate $\int_0^{\frac{\pi}{2}} \frac{\sin^2 x\,dx}{\sin x + \cos x}$.

(3 marks)



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**13.** Let $\overrightarrow{OA} = -\mathbf{i} + \mathbf{j} - \mathbf{k}$, $\overrightarrow{OB} = 3\mathbf{i} + \mu\mathbf{j} + \lambda\mathbf{k}$ and $\overrightarrow{OC} = \mathbf{j} + 2\mathbf{k}$, where $O$ is the origin, and $\lambda$ and $\mu$ are real numbers. It is given that $\angle AOB = \angle BOC = 90°$.

**(a)** Find the values of $\lambda$ and $\mu$. (3 marks)

**(b)** **(i)** Find the volume of the tetrahedron $OABC$.

**(ii)** If $\Delta ABC$ is the base, find the height of the tetrahedron.

**(iii)** Find the angle between the line $OB$ and the plane $ABC$, correct to the nearest 0.1°.

(7 marks)

**(c)** Let $D$ be the foot of perpendicular from $O$ to the plane $ABC$. Sam claims that $D$ lies on the line joining $B$ and the mid-point of $AC$. Do you agree? Explain your answer.

(3 marks)



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