2021-2022 S5
1st TERM EXAM
MATH EP
M2

2021 –

S5 First Term E

MATHEMATICS E

Module 2 (Algebra

Question–Answer Book

12th January, 2022
8:15 am – 10:15 am (2 hours)
This paper must be answered in English

INSTRUCTIONS

1. Write your name, class and class number in the spaces provided on this cover.

2. This paper consists of TWO sections, A and B.

3. Attempt ALL questions in this paper. Write your answers in the spaces provided in this Question – Answer Book. Do not write in the margins. Answers written in the margins will not be marked.

4. Unless otherwise specified, all working must be clearly shown.

5. Unless otherwise specified, numerical answers must be exact.

6. The diagrams in this paper are not necessarily drawn to scale.

Sections	Marks
A Total	**/49**
B Total	**/31**
TOTAL	**/80**

FORMULAS FOR REFERENCE

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$	$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$
$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$	$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$
$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$	$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$
$2\sin A \cos B = \sin(A+B) + \sin(A-B)$	$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$
$2\cos A \cos B = \cos(A+B) + \cos(A-B)$	
$2\sin A \sin B = \cos(A-B) - \cos(A+B)$	

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Section A (49 marks)

1. Find $\dfrac{d}{d\theta}\sec 6\theta$ from first principles. (5 marks)

2. Let m be a positive integer.

(a) Expand $(1+5x)^m(1-ax)^5$ in ascending powers of x up to the x^2 term. (2 marks)

(b) If the coefficients of x and x^2 in the above expansion are 20 and 90 respectively, find a and m. (4 marks)

3. (a) Prove that $4\cos\theta\cos 3\theta\cos 5\theta = \cos\theta + \cos 3\theta + \cos 7\theta + \cos 9\theta$.

(b) Solve $\cos\theta + \cos 3\theta + \cos 7\theta + \cos 9\theta = 0$ for $0 \le \theta \le \frac{\pi}{3}$.

(4 marks)

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4. Prove, by mathematical induction, that

$$\frac{1\times 2}{2\times 3}+\frac{2\times 2^2}{3\times 4}+\frac{3\times 2^3}{4\times 5}+\cdots+\frac{n\times 2^n}{(n+1)(n+2)}=\frac{2^{n+1}}{n+2}-1$$

for all positive integers n.

(5 marks)

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5. Let $f(x)$ be a polynomial. Figure 1 shows a sketch of the curve $y = f'(x)$, where $-2 \le x \le 6$. The curve cuts the x-axis at the origin and $(a, 0)$, where $0 < a < 6$. It is known that the areas of the shaded regions R_1 and R_2 as shown in Figure 1 are 3 and 1 respectively.

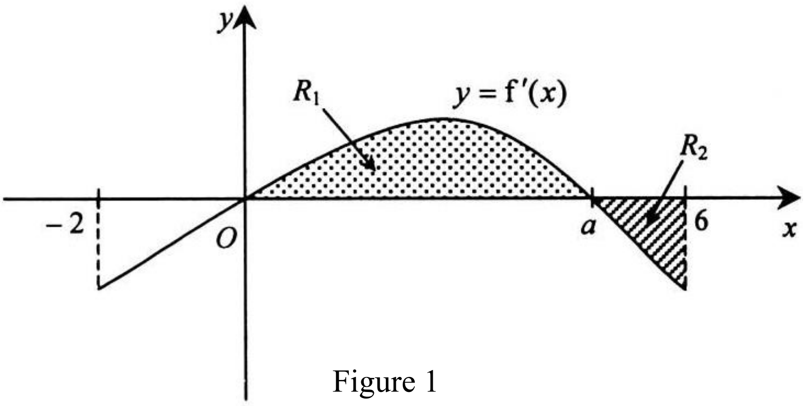


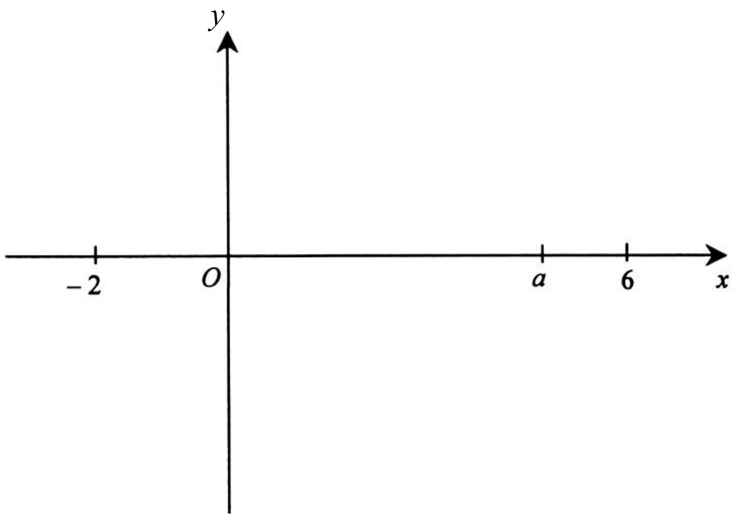
Figure 1

(a) Write down the x-coordinates of the maximum and minimum points of the curve $y = f(x)$ for $-2 \le x \le 6$. (2 marks)

(b) It is known that $f(-2) = 2$ and $f(0) = 1$.

(i) By considering $\int_0^a f'(x)\ dx$, find the value of $f(a)$.

(ii) In Figure 2, sketch the curve $y = f(x)$ for $-2 \le x \le 6$.



(5 marks)

Figure 2

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6. Evaluate $\lim_{x \to 0} \frac{e^{2x} - e^{-2x}}{x}$. (3 marks)

7. Let C be the curve $3e^{x-y} = x^2 + y^2 + 1$. Find the equation of tangent to C at the point $(1,1)$.

(5 marks)

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8. Let A, B and C be three square matrices of the same order. If $B = A^T$ and $C = -B^T$, express $(AB)^T$ in terms of C. (4 marks)

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9.

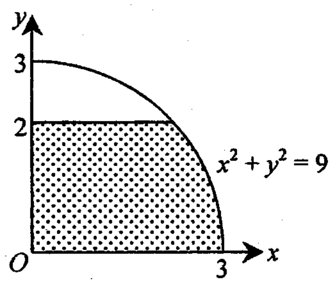

Figure 3

In Figure 3, the shaded region is bounded by the circle $x^2 + y^2 = 9$, the x-axis, the y-axis and the line $y = 2$. Find the volume of the solid generated by revolving the region about the y-axis.

(3 marks)

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10. Let $M = \begin{pmatrix} 2 & 7 \\ -1 & -6 \end{pmatrix}$. Denote the 2×2 identity matrix by I.

(a) Find a pair of real numbers a and b such that $M^2 = aM + bI$. (3 marks)

(b) Prove that $6M^n = \left(1-(-5)^n\right)M + \left(5+(-5)^n\right)I$ for all positive integers n. (4 marks)

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Section B (31 marks)

11. (a) Show that $\frac{d}{dx}\ln(x+\sqrt{x^2+1})=\frac{1}{\sqrt{x^2+1}}$. (3 marks)

(b) Using differentiation, show that $\frac{f(x)}{x}=\int\frac{f'(x)}{x}dx-\int\frac{f(x)}{x^2}dx$. (3 marks)

(c) Using the above results, evaluate $\int\frac{\sqrt{x^2+1}}{x^2}dx$. (3 marks)

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12. In Figure 4, the shaded region is bounded by the curve $x=\sqrt{\sin y+4}$, the straight line $y=3\pi$ and the two axes.

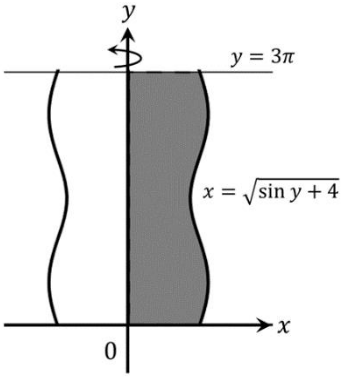

Figure 4

(a) Find the volume of the solid of revolution if the shaded region is revolved about the y-axis. (3 marks)

(b) A glass cup is in the shape of the solid described in (a).

(i) Water is poured into the cup such that the depth of water increases at a constant rate of 0.2 unit/s. When the cup is half full of water, find the rate of change of the volume of water.

(ii) After the cup is fully filled with water, the cup cracks suddenly at the bottom and the water leaks out at a constant rate of π cubic units per second. When the depth of water is two-third of the height of the cup, find the rate of change of water level.

(6 marks)

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13. (a) Using integration by substitution, prove that $\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln\left(\sin\left(\frac{\pi}{4}-x\right)\right) dx = \int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln\left(\sin x\right) dx$.

(3 marks)

(b) Using (a), prove that $\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln(\cot x - 1)\, dx = \frac{\pi \ln 2}{24}$. (3 marks)

(c) (i) Using $\cot\left(A-B\right) = \frac{\cot A \cot B + 1}{\cot B - \cot A}$, or otherwise, prove that $\cot \frac{\pi}{12} = 2+\sqrt{3}$.

(ii) Using integration by parts, prove that $\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \frac{x \csc^2 x}{\cot x - 1}\, dx = \frac{\pi}{8} \ln\left(2+\sqrt{3}\right)$.

(7 marks)

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