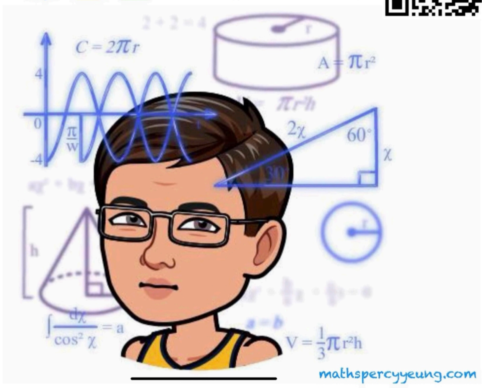
2019-20
MATH EP
F.5 M2



2019 - 2020 2nd Term Examination

Form 5 MATHEMATICS
Extended Part
Module 2 (Algebra and Calculus)

Question–Answer Book

6th July, 2020. (Monday)

8:15 am – 10:45 am (2.5 hours)

This paper must be answered in English.

INSTRUCTIONS

1. After the announcement of the start of the examination, you should first write your name, class and class number in the spaces provided on this cover.
2. This paper consists of Section A and Section B.
3. Answer ALL questions. Write your answers in the spaces provided in this Question-Answer Book.
4. Graph paper and supplementary answer sheets will be supplied on request. Write your name, class, class number and mark the question number box on each sheet.
5. Unless otherwise specified, all working must be clearly shown.
6. Unless otherwise specified, numerical answers must be exact.
7. The diagrams in this paper are not necessarily drawn to scale.

	Marks
Section A	**/ 50**
Section B	**/ 50**
Grand Total	**%**

FORMULAS FOR REFERENCE

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$	$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$
$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$	$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$
$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$	$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$
$2\sin A \cos B = \sin(A+B) + \sin(A-B)$	$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$
$2\cos A \cos B = \cos(A+B) + \cos(A-B)$	
$2\sin A \sin B = \cos(A-B) - \cos(A+B)$	

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Section A (50 marks)

1. Find $\frac{d}{dx}(x\sin x)$ from first principles.

(4 marks)

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2. It is given that $(1-kx)(1+3x)^n = 1+22x+204x^2$ + terms involving higher powers of x, where n is a positive integer. Find the values of k and n. (5 marks)

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3. It is given that $P(-1, 0)$ is a point on $y^2 = x^2 - x\sin y - 1$. Find the equation of the tangent to the curve at P .

(5 marks)

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4. **(a)** If $\tan A = \dfrac{2 - \tan B}{1 - 2\tan B}$, prove that $\sin(A + B) = 2\cos(A - B)$.

(b) Using (a), solve the equation $\tan(x + 50°) = \dfrac{2 - \tan(x - 10°)}{1 - 2\tan(x - 10°)}$, where $0° \le x \le 90°$.

(6 marks)

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5. **(a)** Prove that $\sin^2 x \cos^2 x = \dfrac{1-\cos 4x}{8}$.

(b) Let $f(x) = \sin^4 x + \cos^4 x$.

(i) Express $f(x)$ in the form $A \cos Bx + C$, where A , B and C are constants.

(ii) Solve the equation $8f(x) = 7$, where $0 \le x \le \dfrac{\pi}{2}$.

(7 marks)

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6. **(a)** Using mathematical induction, prove that

$$\sum_{r=1}^{n} r(r+1)^2 = \frac{n(n+1)(n+2)(3n+5)}{12}$$

for all positive integers n .

(b) Hence, evaluate $50 \times 5.1^2 + 51 \times 5.2^2 + 52 \times 5.3^2 + \cdots + 100 \times 10.1^2$.

(8 marks)

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7. **(a)** Let M be a 3×3 matrix such that $M^2 = M^T$, where M^T is the transpose of M.

(i) Prove that $|M| = 0$ or 1.

(ii) Suppose M is a non-singular matrix. Prove that $M^4 = M$ and $M^{-1} = M^T$.

(b) Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$. Using **(a)**, or otherwise, find A^{-1}.

(7 marks)

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8. **(a)** In Figure 1, the shaded region is bounded by the curve $y=\frac{x^2}{4}-1$, the positive x-axis, the positive y-axis and the line $y=h$, where $h>0$. Show that the volume of the solid generated by revolving the shaded region about the y-axis is $2\pi h(h+2)$.

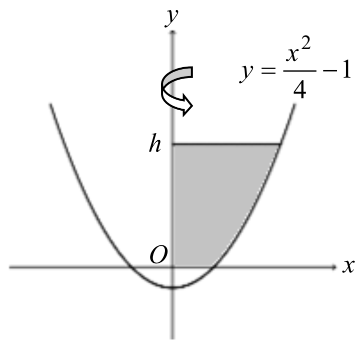

Figure 1

(b) In Figure 2, the bowl is formed by revolving the shaded region about the y-axis as described in **(a)**. The radii of the bottom and the top of the bowl are 2 cm and 6 cm respectively.

(i) Find the height of the bowl.

(ii) If water is poured into the empty bowl at the rate of $\pi\left(1+\frac{t}{4}\right)$ cm^3/s, where t s is the time used for pouring the water into the bowl, when will the water level be half of the height of the bowl?

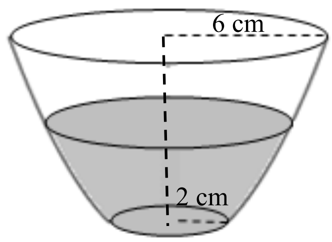

Figure 2

(8 marks)

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Section B (50 marks)

9. Define $f(x)=\dfrac{3x^2-2x-5}{x-2}$ for all $x \neq 2$. Denote the graph of $y=f(x)$ by G.

(a) Find the asymptote(s) of G.

(3 marks)

(b) Find $f'(x)$.

(2 marks)

(c) Find the maximum point(s) and the minimum point(s) of G.

(4 marks)

(d) Find the area of the region bounded by G and the x-axis.

(4 marks)

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10. **(a)** Find $\frac{d}{dx}\ln\frac{2+x}{2-x}$.

(1 mark)

(b) Let $0 \le x \le \frac{\pi}{4}$. Prove that $\frac{1}{3+5\cos 2x} = \frac{\sec^2 x}{10-2\sec^2 x}$.

(1 mark)

(c) Evaluate $\int_0^{\frac{\pi}{4}} \frac{1}{3+5\cos 2x}\,dx$.

(3 marks)

(d) Let $f(x)$ be a continuous function defined on $\mathbf{R}$ such that $f(-x) = -f(x)$ for all $x \in \mathbf{R}$.
Prove that $\int_{-a}^{a} f(x)\ln(1+e^x)\,dx = \int_0^a x\,f(x)\,dx$ for any $a \in \mathbf{R}$.

(4 marks)

(e) Evaluate $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sin 2x}{(3+5\cos 2x)^2}\ln(1+e^x)\,dx$.

(4 marks)

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11. (a) Solve the equation

$$\begin{vmatrix} 3-\lambda & 4 \\ 1 & -\lambda \end{vmatrix} = 0 \qquad \ldots\ldots (*) .$$

(2 marks)

(b) Let λ_1 and λ_2 ($\lambda_1 > \lambda_2$) be the roots of (*). Let $P = \begin{pmatrix} 1 & 1 \\ a & b \end{pmatrix}$. It is given that

$$\begin{pmatrix} 3 & 4 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ a & b \end{pmatrix} = \begin{pmatrix} \lambda_1 & \lambda_2 \\ \lambda_1 a & \lambda_2 b \end{pmatrix} ,$$

where a and b are constants.

(i) Find P .

(ii) Evaluate $P^{-1}\begin{pmatrix} 3 & 4 \\ 1 & 0 \end{pmatrix}P$.

(iii) Using **(b)(ii)**, evaluate $\begin{pmatrix} 3 & 4 \\ 1 & 0 \end{pmatrix}^{10}$.

(8 marks)

(c) A sequence is defined by

$$x_1 = 1 \ , \ x_2 = 1 \ \text{ and } \ x_n = 3x_{n-1} + 4x_{n-2} \ \text{ for } \ n = 3, 4, 5, \ldots \ .$$

It is known that this sequence can be expressed in the matrix form

$$\begin{pmatrix} x_n \\ x_{n-1} \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_{n-1} \\ x_{n-2} \end{pmatrix} .$$

Using the result of **(b)(iii)**, find x_{12} .

(2 marks)

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12. Consider the system of linear equations in real variables x, y, z

$$(E): \begin{cases} x + y - z = a \\ x - y - (\lambda - 1)z = b \\ \lambda x + 3y - 2z = 4a \end{cases},$$

where λ, a and b are real numbers.

(a) Assume that (E) has a unique solution.

(i) Find the range of values of λ.

(ii) Express z in terms of λ, a and b.

(5 marks)

(b) Assume that $\lambda = 1$ and (E) is consistent.

(i) Express b in terms of a.

(ii) Hence, solve (E) in terms of a.

(4 marks)

(c) If (x, y, z) is a real solution of the system of linear equations

$$\begin{cases} x + y - z = 6 \\ x - y = -12 \\ x + 3y - 2z = 24 \end{cases},$$

is $x^2 + y^2 + z^2 > 75$ always true? Explain your answer.

(3 marks)

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